Optimizing the first eigenvalue of some quasilinear operators with respect to the boundary conditions *

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Geometric and Analytic Inequalities

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*Joint work with F. Della Pietra (Napoli Federico II) e H. Kovařík (Brescia)

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Question

To optimize $\ell_1(\sigma, \Omega)$ with respect to the function σ. More precisely, to study the existence and the properties of σ which minimize or maximize $\ell_1(\sigma, \Omega)$, under the constraint

$$\int_{\partial\Omega} \sigma d\mathcal{H}^{n-1} = m > 0.$$

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Proposition (Della Pietra-G.-Kovařík 2015)

For every $\sigma \in L^1(\partial\Omega)$, with $\sigma \geqslant 0$, then:

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- *if $\sigma > 0$ on $\Gamma \subseteq \partial\Omega$ and $\mathcal{H}^{n-1}(\Gamma) > 0$, then $\ell_1(\sigma,\Omega) > 0$;*
- *$\ell_1(\sigma,\Omega)$ is simple, and v has costant sign in Ω.*

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$$\Lambda_1^D(\Omega) \geqslant \frac{C}{R_\Omega^p}, \qquad \Omega \text{ convex}, \ R_\Omega \text{ inradius of } \Omega.$$

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$$p = 2 \qquad \mu(\Omega) \leqslant \mu(B), \qquad |\Omega| = |B|$$

(Szegő 1954 - Weinberger 1956)

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(Szegő 1954 - Weinberger 1956)

$$1<p<+\infty \qquad \mu(\Omega)\leqslant\Lambda_1^D(B), \qquad |\Omega|=|B|,\ \Omega \text{ convex}$$

(Brasco-Nitsch-Trombetti 2015)

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$$\ell_1(\bar{\sigma}, \Omega) \geqslant \left(\frac{p-1}{p}\right)^p \frac{\bar{\sigma}}{R_\Omega \left(1 + \bar{\sigma}^{\frac{1}{p-1}} R_\Omega\right)^{p-1}},$$

(Ω convex, R_Ω inradius of Ω, Kovařík 2012 ($p = 2$), Della Pietra-G., 2014)

Optimization of $\ell_1(\sigma, \Omega)$ with respect to Ω, and σ constant

$\bar{\sigma} < 0$: case $p = 2$

Conjecture (Bareket 1977)

$$\ell_1(\bar{\sigma}, \Omega) \leqslant \ell_1(\bar{\sigma}, B),$$

where B is a ball with $|B| = |\Omega|$.

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In general, the conjecture is false: $\forall n \geqslant 2$ there exist $\bar{\sigma}$: $|\bar{\sigma}| >> 1$ and a spherical shell G, $|G| = |B|$ such that

$$\ell_1(\bar{\sigma}, G) > \ell_1(\bar{\sigma}, B) \qquad \text{(Freitas - Krejčiřík 2015);}$$

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however, the ball is a local maximum (Ferone - Nitsch - Trombetti 2015)

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To optimize $\ell_1(\sigma, \Omega)$ with respect to $\sigma \in \Sigma_m$:

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Upper bound for $\ell_1(\sigma, \Omega)$

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$$\text{then} \qquad \ell_1(\sigma,\Omega) \leqslant \min\left\{\Lambda_1^D(\Omega), \frac{m}{|\Omega|}\right\},$$

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is the first Dirichlet eigenvalue of $-\Delta_p$ in Ω.

Optimizing $\ell_1(\sigma, \Omega)$ with respect to σ: the supremum

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Let $n \geqslant 1$. For any $m > 0$,

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- *σ_m can be explicitly characterized.*

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Proposition

Let $p > 1$, $m > 0$, $\hat{\sigma} \in \Sigma_m(\partial\Omega)$. If $\hat{u} \in W^{1,p}(\Omega)$ is such that

$$\ell_1(\hat{\sigma}, \Omega) = \mathcal{Q}[\hat{\sigma}, \hat{u}] = \frac{\displaystyle\int_\Omega |\nabla \hat{u}|^p dx + \int_{\partial\Omega} \hat{\sigma} |\hat{u}|^p d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |\hat{u}|^p dx},$$

and $\hat{u}$ is constant on $\partial\Omega$, then

$$\Lambda(m, \Omega) = \ell_1(\hat{\sigma}, \Omega).$$

Optimizing $\ell_1(\sigma, \Omega)$ with respect to σ: the supremum

$$\Lambda(m, \Omega) = \sup_{\sigma \in \Sigma_m(\partial\Omega)} \ell_1(\sigma, \Omega)$$

Proof

Let $\hat{\sigma} \in \Sigma_m(\partial\Omega)$ be such that $\ell_1(\hat{\sigma}) = \mathcal{Q}[\hat{\sigma}, \hat{u}]$, with $\hat{u}$ constant on $\partial\Omega$.

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Proof

Let $\hat{\sigma} \in \Sigma_m(\partial\Omega)$ be such that $\ell_1(\hat{\sigma}) = \mathcal{Q}[\hat{\sigma}, \hat{u}]$, with $\hat{u}$ constant on $\partial\Omega$. Then for every $\sigma \in \Sigma_m(\partial\Omega)$ we have:

$$\ell_1(\sigma, \Omega) = \min_{\substack{u \in W^{1,p}(\Omega) \\ u \neq 0}} \mathcal{Q}[\sigma, u] \leqslant \mathcal{Q}[\sigma, \hat{u}] = \frac{\displaystyle\int_\Omega |\nabla \hat{u}|^p dx + \int_{\partial\Omega} \sigma(x) \hat{u}^p d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega \hat{u}^p dx}$$

$$= \frac{\displaystyle\int_\Omega |\nabla \hat{u}|^p dx + m\ \hat{u}|_{\partial\Omega}^p}{\displaystyle\int_\Omega \hat{u}^p dx} = \mathcal{Q}[\hat{\sigma}, \hat{u}] = \ell_1(\hat{\sigma}, \Omega).$$

Hence $\qquad \Lambda(m, \Omega) = \ell_1(\hat{\sigma}, \Omega).$

To prove the existence of $\hat{\sigma}$ for every fixed $\xi \in [0, \Lambda_1^D(\Omega)[$, let $u_\xi \in W_0^{1,p}(\Omega)$ be the unique positive function in Ω which solves

$$(P_{aux}) \qquad \begin{cases} -\Delta_p u_\xi &= \left(\xi^{\frac{1}{p-1}} u_\xi + 1\right)^{p-1} \quad \text{in } \Omega, \\ u_\xi &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

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Let $F : [0, \Lambda_1^D(\Omega)[\to [0, +\infty[$ be the following function

$$F(\xi) = \xi \int_\Omega \left(\xi^{\frac{1}{p-1}} u_\xi + 1\right)^{p-1} dx.$$

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Lemma

The function F is strictly increasing, and $F(\xi) \to +\infty$ for $\xi \to \Lambda_1^D(\Omega)$

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The function F is strictly increasing, and $F(\xi) \to +\infty$ for $\xi \to \Lambda_1^D(\Omega)$

Hence we can define $\xi : [0, \infty[\to [0, \Lambda_1^D(\Omega)[$ as follows:

$$\xi(m) = \xi_m := F^{-1}(m).$$

For any $m > 0$ there exists a unique $u_{\xi_m} > 0$ which solves (P_{aux}) for $\xi = \xi_m$.

$$(P_{aux}) \qquad \begin{cases} -\Delta_p u_\xi &= \left(\xi^{\frac{1}{p-1}} u_\xi + 1\right)^{p-1} \quad \text{in} \;\; \Omega, \\ u_\xi &= 0 \qquad\qquad\qquad\quad \text{on} \;\; \partial\Omega. \end{cases}$$

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Theorem (Della Pietra-G.-Kovařík 2015)

For any $m > 0$, the value $\Lambda(m,\Omega) = \sup_{\sigma \in \Sigma_m(\partial\Omega)} \ell_1(\sigma,\Omega)$ is achieved and

$$\Lambda(m,\Omega) = \ell_1(\sigma_m,\Omega) = \xi_m, \qquad \sigma_m = -\xi_m \, |\nabla u_{\xi_m}|^{p-2} \, \frac{\partial u_{\xi_m}}{\partial \nu} .$$

where u_{ξ_m} is the unique solution to (P_{aux}) with $\xi = \xi_m$ and σ_m is unique.

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If Ω is a ball, then the unique positive solution to (P_{aux}) is radial. Then in this case σ_m is constant:

$$\sigma_m = \frac{m}{|\partial\Omega|}.$$

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Proposition

The maximum $\Lambda(m, \Omega)$ verifies the following Faber-Krahn inequality

$$\Lambda(m, \Omega) \geqslant \Lambda(m, B_R),$$

where B_R is a ball such that $|\Omega| = |B_R|$.

Optimizing $\ell_1(\sigma, \Omega)$ with respect to σ: the infimum

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The behavior of $\lambda(m, \Omega)$ depends on p and on the dimension n.

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Case $p \leqslant n$

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Case $p \leqslant n$

Theorem (Della Pietra-G.-Kovařík '15)

If $1 < p \leqslant n$, then for every $m > 0$ we have

$$\lambda(m, \Omega) = 0$$

and the infimum is not achieved.

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If $p > n$, then for every $m > 0$ we have

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Moreover, $\lambda(m, \Omega)$ is a minimum iff $n = 1$.

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If $p = 2$, $\lambda(m, \Omega) = 0$ $\forall n \geqslant 2$, and for $n = 1$ $\lambda(\sigma, \Omega) > 0$, is achieved.

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For $x_0 \in \partial\Omega$ fixed, and $\forall j \in \mathbb{N}$ let

$$\sigma_j(x) = \begin{cases} \alpha_j & \text{if } x \in B_{2^{-j}}(x_0) \cap \partial\Omega, \\ 0 & \text{otherwise,} \end{cases}$$

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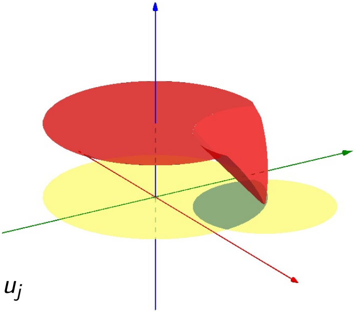

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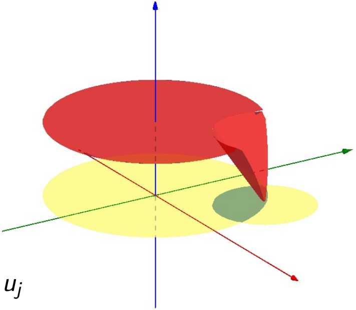

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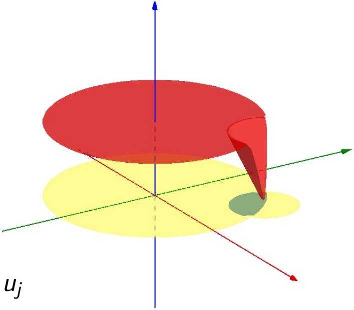

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$$u_j(x) = \begin{cases} j|x - x_0| & \text{in } B_{\frac{1}{j}}(x_0) \cap \Omega, \\ 1 & \text{in } \Omega \setminus B_{\frac{1}{j}}(x_0). \end{cases}$$

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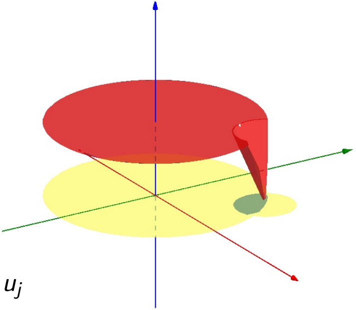

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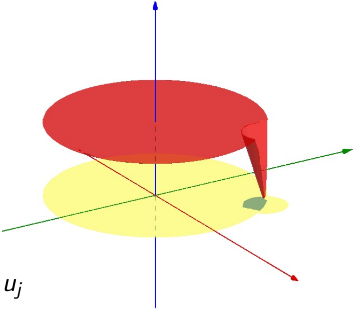

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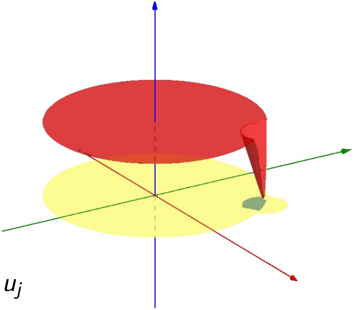

$$0 \leqslant \ell_1(\sigma_j, \Omega) \leqslant$$

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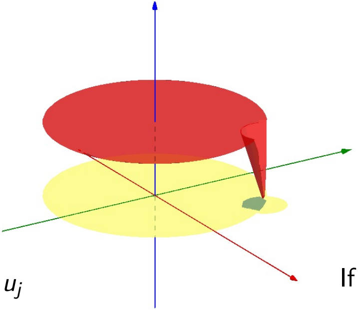

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If $p = n$, $u_j = -\log j / \log(|x - x_0|)$ per $x \sim x_0$

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If $p > n$, *then* $\lambda(m, \Omega) > 0$.

Moreover, $\lambda(m, \Omega)$ *is a minimum iff* $n = 1$.

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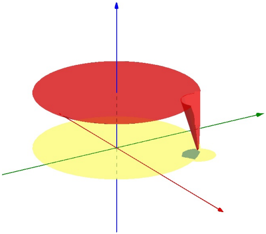

The previous sequence $(\sigma_j,\, u_j)$ does not give any information:

$$\mathcal{Q}[\sigma_j, u_j] \leqslant \frac{j^p \left|B_{\frac{1}{j}}(x_0)\right| + j^p\, 2^{-jp}\, m}{|\Omega| - \left|B_{\frac{1}{j}}(x_0)\right|} \to +\infty$$

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If $n = 1$, $\Omega =]a, b[$, $\lambda_1(\sigma, \Omega)$ is achieved

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where $\sigma_a, \sigma_b \in \Sigma_m(\{a, b\})$ are such that
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Relaxed problem and concentration effect ($p > n$)

$$\ell_1(\mu, \Omega) = \inf_{u \in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} |u|^p \, d\mu}{\displaystyle\int_\Omega |u|^p dx}, \qquad \mu \in \mathcal{M}(m),$$

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In general, x_m is not unique; his position depends on m and on Ω.

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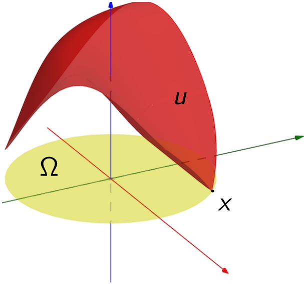

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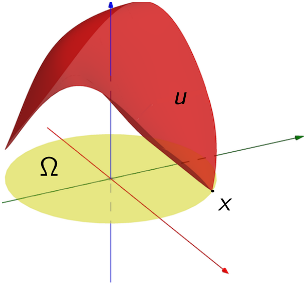

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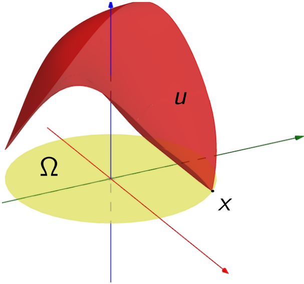

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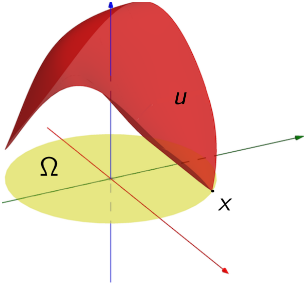

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$$\lambda_1(\Omega) := \min\{\lambda_1(x;\Omega);\ x \in \partial\Omega\}.$$

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For every $m > 0$, let $(\bar{u}_m, \mu_m)$ a minimum for $\lambda(m, \Omega)$, with $\|\bar{u}_m\|_p = 1$:

$$\lambda(m, \Omega) = \mathcal{Q}[\mu_m, \bar{u}_m] = \int_\Omega |\nabla \bar{u}_m|^p \, dx + \int_{\partial\Omega} \bar{u}_m^p d\mu_m,$$

with $\mu_m = m\delta_{x_m}$, Dirac measure concentrated in $x_m \in \partial\Omega$.

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$$\begin{matrix} \bar u_m(x_m)\to 0 \text{ per } m\to+\infty \\ \bar u_m \rightharpoonup \bar u\in W^{1,p}(\Omega)\cap C(\Omega)\end{matrix} \Rightarrow \begin{cases} \|\bar u\|_p = 1 \\ \bar u(\bar x)=0, \quad \text{where } x_m\to\bar x\end{cases}$$

Hence $\bar u$ is an admissible test function for $\lambda_1(\bar x;\Omega)$, and

$$\lambda_1(\Omega) = \liminf_{m\to\infty}\lambda(m,\Omega) \geqslant \liminf_{m\to\infty}\int_\Omega |\nabla \bar u_m|^p\,dx \geqslant \int_\Omega|\nabla\bar u|^p\,dx$$
$$\geqslant \lambda_1(\bar x;\Omega) \geqslant \min_{x\in\partial\Omega}\lambda_1(\bar x;\Omega) = \lambda_1(\Omega).$$

Estimates on $\lambda(m, \Omega)$ and $\Lambda(m, \Omega)$

$$\lambda(m, \Omega) = \inf_{\sigma \in \Sigma_m(\partial\Omega)} \ell_1(\sigma, \Omega), \qquad \Lambda(m, \Omega) = \sup_{\sigma \in \Sigma_m(\partial\Omega)} \ell_1(\sigma, \Omega)$$

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Proposition (Lower bound for Λ; Della Pietra-G.-Kovařík '15)

For every $p>1$ and $m>0$ we have

$$\Lambda(m,\Omega) \geqslant \frac{m\,\Lambda_1^D(\Omega)}{\left[\left(|\Omega|\,\Lambda_1^D(\Omega)\right)^{1/(p-1)} + m^{1/(p-1)}\right]^{p-1}},$$

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- $\lim_{m\to 0} \lambda(m,\Omega) = \lim_{m\to 0} \Lambda(m,\Omega) = 0$

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The bounds on $\Lambda(m, \Omega; p)$ imply

$$\lim_{p \to 1} \Lambda(m, \Omega; p) = \min \left\{ \frac{m}{|\Omega|}, h(\Omega) \right\}.$$

Thanks for the attention!