Isoperimetric inequalities for complete proper minimal submanifolds in hyperbolic space

(joint work with Sung-Hong Min)

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Introduction

Theorem (Classical Isoperimetric Inequality)

Let C be a simple closed curve in the plane whose length is L and that encloses an area A. Then

$$4\pi A \leqslant L^2.$$

Equality holds if and only if C is a circle.

Two ways of a generalization

- $\Omega^2 \subset \mathbb{R}^2 \to \Omega^n \subset \mathbb{R}^n$

$$n^n \omega_n |\Omega|^{n-1} \leqslant |\partial\Omega|^n$$

  and equality holds if and only if Ω is a ball. (ω_n is the volume of the unit ball in $\mathbb{R}^n$)

- For $M^n \subset \bar{M}^{n+m}$
  - Simplest case: $M^2 \subset \mathbb{R}^2 \subset \mathbb{R}^3$. In this case we have $4\pi A \leqslant L^2$ with equality if and only if M is a disk.
  - A natural extension: a minimal surface $M^2 \subset \mathbb{R}^3$

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Isoperimetric inequality for compact minimal surfaces

Question

For a minimal surface $M^2 \subset \mathbb{R}^3$, does the isoperimetric inequality

$$4\pi A \leqslant L^2.$$

hold? And does equality hold if and only if the minimal surface is a disk?

This problem has been partially proved but not completely.

- Yes, for simply-connected case. (Carleman 1921)
- Yes, for doubly-connected case. (Osserman-Schiffer 1975, Feinberg 1977)
- Yes, for $\sharp(\partial M) \leqslant 2$. (Li-Schoen-Yau 1984, Choe 1990)
- Yes, for triply-connected case. (Choe-Schoen, recent)

This problem is still open.

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Two partial answers:

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In hyperbolic space

Two important results about isoperimetric inequalities on minimal submanifolds in hyperbolic space

- 2-dimensional case: (Choe-Gulliver 1992) For Σ with $\sharp(\partial\Sigma) \leqslant 2$,

$$4\pi \mathrm{Area}(\Sigma) \leqslant \mathrm{Length}(\partial\Sigma)^2 - \mathrm{Area}(\Sigma)^2$$

  with equality if and only if Σ is a geodesic ball in a totally geodesic 2-plane in $\mathbb{H}^n$.

- higher-dimensional case: (Yau 1975, Choe-Gulliver 1992)

$$(k-1)\mathrm{Vol}(\Sigma) \leqslant \mathrm{Vol}(\partial\Sigma).$$

  This is called a linear isoperimetric inequality. However it is not sharp.

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Setting and notations

- $\mathbb{H}^n$: the hyperbolic n-space of constant curvature -1
- We choose the Poincaré ball model B^n among several models of $\mathbb{H}^n$.
- Then B^n can be regarded as both
  - the unit ball in $\mathbb{R}^n$
    and
  - the Poincaré ball model of $\mathbb{H}^n$.
- $ds^2_{\mathbb{H}}$: the hyperbolic metric on B^n
  $ds^2_{\mathbb{R}}$: the Euclidean metric on B^n
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- Note that $\Sigma \subset \mathbb{H}^n$ is called proper if, for any compact subset $C \subset \mathbb{H}^n$, the intersection $C \cap \Sigma$ is also compact in $\mathbb{H}^n$.
- The existence of complete minimal submanifolds in hyperbolic space was proved by M. Anderson(1982, 1983) and F.H. Lin(1989). More precisely, given $\gamma^{k-1} \subset \partial_\infty \mathbb{H}^n$, there exists a k-dimensional area-minimizing Σ satisfying that $\partial\Sigma$.
- $\mathrm{Vol}_{\mathbb{R}}(\Sigma)$: the k-dimensional Euclidean volume of Σ
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Linear isoperimetric inequality implies classical isoperimetric inequality

Theorem

Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n. Then

$$\mathrm{Vol}_{\mathbb{R}}(\Sigma) \leqslant \frac{1}{k}\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma),$$

where equality holds if and only if Σ is a k-dimensional unit ball B^k in B^n.

Corollary

Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n. If $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma) \geqslant \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1}) = k\omega_k$, then

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The conclusion of Corollary is sharp in the following sense: Assume that Σ is totally geodesic in the Poincaré ball model B^n. Since the Euclidean projection of Σ onto the flat hypersurface containing $\partial\Sigma$ is volume-decreasing, we have the reverse isoperimetric inequality

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- Step1:
  Let $r(x)$ and $\rho(x)$ be the Euclidean and hyperbolic distance from the origin to $x \in B^n$, respectively. Recall that the distance functions r and ρ satisfy that

$$\rho = \ln \frac{1+r}{1-r} \quad \text{and} \quad r = \tanh \frac{\rho}{2} = \frac{\sinh \rho}{1+\cosh \rho}.$$

  Denote by B_R the Euclidean ball of radius R centered at the origin for $0 < R < 1$. Note that the Euclidean ball B_R can be thought of as the hyperbolic ball B_{R^*} of radius R^* in the Poincaré ball model B^n, where $R^* = \ln \frac{1+R}{1-R}$.

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- We now consider two kinds of the intersection of Σ with the Euclidean ball B_R and the hyperbolic ball B_{R^*} as following.

Σ_R: the intersection $\Sigma \cap B_R$ (possibly empty) which has the volume form $dV_{\mathbb{R}}$ induced from the Euclidean metric
$\widetilde{\Sigma}_{R^*}$: the intersection $\Sigma \cap B_{R^*}$ which has the volume form $dV_{\mathbb{H}}$ induced from the hyperbolic metric.
Since

$$dV_{\mathbb{R}} = \left(\frac{1-r^2}{2}\right)^k dV_{\mathbb{H}},$$

the Euclidean volume $\mathrm{Vol}_{\mathbb{R}}(\Sigma_R)$ can be computed as

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- Step2: Analyze a very special function of distance
  - Let Σ be a k-dimensional minimal submanifold in $\mathbb{H}^n$. Then the distance ρ satisfies that

$$\triangle_\Sigma \rho = \coth \rho (k - |\nabla_\Sigma \rho|^2).$$

  - Let f be a smooth function in ρ on Σ.

$$\begin{aligned}
\triangle_\Sigma f &= \mathrm{div}(\nabla_\Sigma f) \\
&= f'' |\nabla_\Sigma \rho|^2 + f' \triangle_\Sigma \rho \\
&= f'' |\nabla_\Sigma \rho|^2 + f' \coth \rho (k - |\nabla_\Sigma \rho|^2) \\
&= k f' \coth \rho - |\nabla_\Sigma \rho|^2 (f' \coth \rho - f'').
\end{aligned}$$

- Step2: Analyze a very special function of distance
  - Let Σ be a k-dimensional minimal submanifold in $\mathbb{H}^n$. Then the distance ρ satisfies that

$$\triangle_\Sigma \rho = \coth\rho(k - |\nabla_\Sigma \rho|^2).$$

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Choose a nice function of distance f on $\Sigma \subset B^n$ by

$$f = -\frac{1}{k(k-1)} \cdot \frac{1}{(1+\cosh \rho)^{k-1}}.$$

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Combining the above relations and using divergence theorem, we get

$$\begin{aligned}
\mathrm{Vol}_{\mathbb{R}}(\Sigma_R) &= \int_{\widetilde{\Sigma}_{R*}} \frac{1}{(1+\cosh\rho)^k} dV_{\mathbb{H}} \\
&\leqslant \int_{\widetilde{\Sigma}_{R*}} kf' \coth\rho - (f' \coth\rho - f'') dV_{\mathbb{H}} \\
&\quad + \int_{\widetilde{\Sigma}_{R*}} (1 - |\nabla_\Sigma \rho|^2)(f' \coth\rho - f'') dV_{\mathbb{H}} \\
&= \int_{\widetilde{\Sigma}_{R*}} \triangle_\Sigma f dV_{\mathbb{H}} \\
&= \int_{\partial\widetilde{\Sigma}_{R*}} f' \frac{\partial\rho}{\partial\nu} d\sigma_{\mathbb{H}},
\end{aligned}$$

where $d\sigma_{\mathbb{H}}$ denotes the volume form of the boundary $\partial\widetilde{\Sigma}_{R^*}$ induced from the volume form $dV_{\mathbb{H}}$ of $\widetilde{\Sigma}_{R^*}$ and ν denotes the outward unit conormal vector.

Use

$$f' = \frac{1}{k}\frac{\sinh\rho}{(1+\cosh\rho)^k} \qquad , \qquad \frac{\partial\rho}{\partial\nu} \leqslant 1,$$

and

$$d\sigma_{\mathbb{H}} = \left(\frac{\sinh\rho}{r}\right)^{k-1} d\sigma_{\mathbb{R}},$$

where $d\sigma_{\mathbb{R}}$ denotes the volume form of the boundary $\partial\Sigma_R$ in Euclidean space.

$$
\begin{aligned}
\mathrm{Vol}_{\mathbb{R}}(\Sigma_R) &\leqslant \int_{\partial \widetilde{\Sigma}_{R^*}} f' \frac{\partial \rho}{\partial \nu} d\sigma_{\mathbb{H}} \\
&\leqslant \int_{\partial \widetilde{\Sigma}_{R^*}} \frac{1}{k} \frac{\sinh \rho}{(1+\cosh \rho)^k} d\sigma_{\mathbb{H}} \\
&= \int_{\partial \Sigma_R} \frac{1}{k} \left(\frac{\sinh \rho}{1+\cosh \rho} \right)^k \frac{d\sigma_{\mathbb{R}}}{r^{k-1}} \\
&= \int_{\partial \Sigma_R} \frac{r}{k} d\sigma_{\mathbb{R}} \\
&= \frac{R}{k} \mathrm{Vol}_{\mathbb{R}}(\partial \Sigma_R).
\end{aligned}
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Therefore, letting R tend to 1, we obtain

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\mathrm{Vol}_{\mathbb{R}}(\Sigma) \leqslant \frac{1}{k} \mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma).
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- Step3: Equality case
  It follows from the above inequality that for $0 < R < 1$

$$\frac{R}{k}\mathrm{Vol}_{\mathbb{R}}(\partial\Sigma_R) - \mathrm{Vol}_{\mathbb{R}}(\Sigma_R) \geqslant \int_{\widetilde{\Sigma}_{R*}} (1 - |\nabla_\Sigma \rho|^2) \frac{\sinh^2 \rho}{(1+\cosh \rho)^{k+1}} dV_{\mathbb{H}}.$$

  Thus equality holds in the above inequality if and only if Σ is a cone in B^n, which is equivalent to that Σ is totally geodesic in B^n and contains the origin. Therefore equality holds if and only if Σ is a k-dimensional unit ball B^k centered at the origin in B^n.

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A sharp lower bound for $\mathrm{Vol}_{\mathbb{R}}(\Sigma)$

- (Fraser and Schoen, 2011)
  If Σ is a minimal surface in the unit ball $B^n \subset \mathbb{R}^n$ with (nonempty) boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$, then
  $$\mathrm{Area}(\Sigma) \geqslant \pi.$$
- (Brendle, 2012)
  If Σ is a k-dimensional minimal submanifold in the unit ball B^n and if Σ meets the boundary ∂B^n orthogonally, then
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Question

Do we have an analogue of the above theorems for complete proper minimal submanifolds in B^n?

Theorem

Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n. If Σ contains the origin in B^n, then

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Containing the origin implies the isoperimetric inequality

Corollary

Let Σ be a k-dimensional complete proper minimal submanifold containing the origin in the Poincaré ball model B^n. Then

$$k^k \omega_k \mathrm{Vol}_{\mathbb{R}}(\Sigma)^{k-1} \leqslant \mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)^k,$$

where equality holds if and only if Σ is a k-dimensional unit ball B^k in B^n.

Proof:

$$\begin{aligned}\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma) &\geqslant k\mathrm{Vol}_{\mathbb{R}}(\Sigma)\\ &\geqslant k\mathrm{Vol}_{\mathbb{R}}(B^k)\\ &= \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1})\end{aligned}$$

Since $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma) \geqslant \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1}) = k\omega_k$, we get the conclusion. QED

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Monotonicity implies the sharp lower bound for $\mathrm{Vol}_{\mathbb{R}}(\Sigma)$

Theorem

Let Σ be a k-dimensional complete minimal submanifold in B^n. Then the function $\dfrac{\mathrm{Vol}_{\mathbb{R}}(\Sigma \cap B_r)}{r^k}$ is nondecreasing in r for $0 < r < 1$. In other words,

$$\frac{d}{dr}\left(\frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma \cap B_r)}{r^k}\right) \geqslant 0,$$

which is equivalent to

$$\frac{d}{d\rho}\left(\frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma \cap B_r)}{r^k}\right) \geqslant 0.$$

Recall that the density $\Theta(\Sigma, p)$ of a k-dimensional submanifold Σ in a Riemannian manifold M at a point $p \in M$ is defined to be

$$\Theta(\Sigma, p) = \lim_{\varepsilon \to 0} \frac{\mathrm{Vol}(\Sigma \cap B_\varepsilon(p))}{\omega_k \varepsilon^k},$$

where $B_\varepsilon(p)$ is the geodesic ball of M with radius ε and center p. As a consequence of monotonicity theorem, we have

Corollary

Let Σ be a k-dimensional complete proper minimal submanifold containing the origin in the Poincaré ball model B^n. Then

$$\mathrm{Vol}_{\mathbb{R}}(\Sigma) \geqslant \omega_k = \mathrm{Vol}_{\mathbb{R}}(B^k).$$

Proof.

Since the function $\frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma_r)}{r^k}$ is nondecreasing in r by monotonicity,

$$\mathrm{Vol}_{\mathbb{R}}(\Sigma) = \lim_{r \to 1-} \frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma_r)}{r^k} \geqslant \lim_{r \to 0+} \frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma_r)}{r^k} = \omega_k \Theta(\Sigma, O) \geqslant \omega_k.$$

□

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Möbius volume

Definition

Let Γ be a compact submanifold of $\mathbb{S}^{n-1}$. Let $\text{Möb}(\mathbb{S}^{n-1})$ be the group of all Möbius transformations of $\mathbb{S}^{n-1}$. The Möbius volume $\mathrm{Vol}_{\mathrm{M}}(\Gamma)$ of Γ is defined to be

$$\mathrm{Vol}_{\mathrm{M}}(\Gamma) = \sup\{\mathrm{Vol}_{\mathbb{R}}(g(\Gamma))\ g \in \text{Möb}(\mathbb{S}^{n-1})\}.$$

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  Let Γ be a k-dimensional compact submanifold of $\mathbb{S}^{n-1}$. Then

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Better lower bound for the $\mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma)$

Theorem

Let Σ be a k-dimensional complete proper minimal submanifold in B^n. Then

$$\mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma) \geqslant \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1}) \cdot \max_{p \in \Sigma} \Theta(\Sigma, p).$$

Proof.

Since Σ is proper in B^n, $\max_{p\in\Sigma}\Theta(\Sigma,p)$ is finite. Moreover the maximum is attained in Σ because the density $\Theta(\Sigma,p)$ is integer-valued there. Now we may assume that $\max_{p\in\Sigma}\Theta(\Sigma,p)$ is attained at $q\in\Sigma$. Take an isometry φ of hyperbolic space $\mathbb{H}^n$ such that $\varphi(q)=O$. Since the group of all isometries of $\mathbb{H}^n$ is isomorphic to $\mathrm{M\ddot{o}b}(\mathbb{S}^{n-1})$, we may consider φ as an element of $\mathrm{M\ddot{o}b}(\mathbb{S}^{n-1})$. Then

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where we used the invariance of the density under an isometry of hyperbolic space in the last equality. This completes the proof. □

Remark

Since $\max_{p\in\Sigma}\Theta(\Sigma,p)\geqslant 1$, this theorem gives another proof of theorem by Min for $\Gamma=\partial_\infty\Sigma$.

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Using this concept, we obtain an isoperimetric inequality for any complete proper minimal submanifold in hyperbolic space with no assumption on Σ unlike the previous results.

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Proof

From the linear isoperimetric inequality, it follows that for any isometry φ of $\mathbb{H}^n$,

$$\mathrm{Vol}_{\mathbb{R}}(\varphi(\Sigma)) \leqslant \frac{1}{k}\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \varphi(\Sigma)).$$

Therefore by the definition of the Möbius volume

$$\mathrm{Vol}_{\mathrm{M}}(\Sigma) \leqslant \frac{1}{k}\mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma).$$

It follows from the previous theorem that

$$\mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma) \geqslant \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1}).$$

Proof

Therefore

$$\begin{aligned} k^k \omega_k \mathrm{Vol}_{\mathrm{M}}(\Sigma)^{k-1} &\leqslant k^k \omega_k \left(\frac{1}{k} \mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma) \right)^{k-1} \\ &= k\omega_k \mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma)^{k-1} \\ &= \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1}) \mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma)^{k-1} \\ &\leqslant \mathrm{Vol}_{\mathrm{M}}(\partial_\infty \Sigma)^k, \end{aligned}$$

which completes the proof.

Remark

We see that if Σ is a k-dimensional complete totally geodesic submanifold in B^n, then equality holds in the inequality. However we do not know whether the converse is true.

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Thank you.