A Yang-Baxter Equation for Metaplectic Ice

From Whittaker Functions to Quantum Groups

Ben Brubaker Valentin Buciumas Daniel Bump

July 22, 2016

- Let $\widetilde{GL}(r, F)$ be the n-fold metaplectic cover of $\mathrm{GL}(r)$. Whittaker models are not unique if $n > 1$.

- Let $\widetilde{GL}(r, F)$ be the n-fold metaplectic cover of $\mathrm{GL}(r)$. Whittaker models are not unique if $n > 1$.
- In 2012 Brubaker, Bump, Chinta, Friedberg and Gunnells proposed statistical mechanical systems whose partition functions are spherical Whittaker functions.

- Let $\widetilde{GL}(r, F)$ be the n-fold metaplectic cover of $\mathrm{GL}(r)$. Whittaker models are not unique if $n > 1$.
- In 2012 Brubaker, Bump, Chinta, Friedberg and Gunnells proposed statistical mechanical systems whose partition functions are spherical Whittaker functions.
- If $n = 1$, then taking into account prior work of Casselman-Shalika and Tokuyama, such systems were found by Hamel and King.

- Let $\widetilde{GL}(r, F)$ be the n-fold metaplectic cover of $\mathrm{GL}(r)$. Whittaker models are not unique if $n > 1$.
- In 2012 Brubaker, Bump, Chinta, Friedberg and Gunnells proposed statistical mechanical systems whose partition functions are spherical Whittaker functions.
- If $n = 1$, then taking into account prior work of Casselman-Shalika and Tokuyama, such systems were found by Hamel and King.
- If $n = 1$, Brubaker, Bump and Friedberg proved the formulas of Tokuyama and Hamel and King using the Yang-Baxter equation.

- Let $\widetilde{GL}(r, F)$ be the n-fold metaplectic cover of $\mathrm{GL}(r)$. Whittaker models are not unique if $n > 1$.
- In 2012 Brubaker, Bump, Chinta, Friedberg and Gunnells proposed statistical mechanical systems whose partition functions are spherical Whittaker functions.
- If $n = 1$, then taking into account prior work of Casselman-Shalika and Tokuyama, such systems were found by Hamel and King.
- If $n = 1$, Brubaker, Bump and Friedberg proved the formulas of Tokuyama and Hamel and King using the Yang-Baxter equation.
- BBCFG described two other ways the Yang-Baxter equation could be applied to p-adic Whittaker functions if available.

- Let $\widetilde{GL}(r, F)$ be the n-fold metaplectic cover of $\mathrm{GL}(r)$. Whittaker models are not unique if $n > 1$.
- In 2012 Brubaker, Bump, Chinta, Friedberg and Gunnells proposed statistical mechanical systems whose partition functions are spherical Whittaker functions.
- If $n = 1$, then taking into account prior work of Casselman-Shalika and Tokuyama, such systems were found by Hamel and King.
- If $n = 1$, Brubaker, Bump and Friedberg proved the formulas of Tokuyama and Hamel and King using the Yang-Baxter equation.
- BBCFG described two other ways the Yang-Baxter equation could be applied to p-adic Whittaker functions if available.
- Until now this powerful tool has until now been unavailable if $n > 1$.

Fix Langlands parameters $z_1, \cdots, z_r$ and a corresponding representation of $\widetilde{GL}(r, F)$. Let λ be a partition. Let $p \in F$ be prime and

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_r} \end{pmatrix}$$

embedded in $\widetilde{GL}(r, F)$ by a standard section. We will describe a formula for the value $W(p^\lambda)$, one particular spherical Whittaker function for the representation index by z_i.

Fix Langlands parameters $z_1, \cdots, z_r$ and a corresponding representation of $\widetilde{GL}(r, F)$. Let λ be a partition. Let $p \in F$ be prime and

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_r} \end{pmatrix}$$

embedded in $\widetilde{GL}(r, F)$ by a standard section. We will describe a formula for the value $W(p^\lambda)$, one particular spherical Whittaker function for the representation index by z_i.

- We begin by recalling the ice-type models that were the second solvable lattice model to be studied.

Fix Langlands parameters $z_1, \cdots, z_r$ and a corresponding representation of $\widetilde{GL}(r, F)$. Let λ be a partition. Let $p \in F$ be prime and

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_r} \end{pmatrix}$$

embedded in $\widetilde{GL}(r, F)$ by a standard section. We will describe a formula for the value $W(p^\lambda)$, one particular spherical Whittaker function for the representation index by z_i.

- We begin by recalling the ice-type models that were the second solvable lattice model to be studied.
- After Onsager solved the Ising model, Lieb, Sutherland and Baxter solved the field-free six-vertex model.

Fix Langlands parameters $z_1, \cdots, z_r$ and a corresponding representation of $\widetilde{GL}(r, F)$. Let λ be a partition. Let $p \in F$ be prime and

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_r} \end{pmatrix}$$

embedded in $\widetilde{GL}(r, F)$ by a standard section. We will describe a formula for the value $W(p^\lambda)$, one particular spherical Whittaker function for the representation index by z_i.

- We begin by recalling the ice-type models that were the second solvable lattice model to be studied.
- After Onsager solved the Ising model, Lieb, Sutherland and Baxter solved the field-free six-vertex model.
- In this context, Baxter employed the Yang-Baxter equation in an ingenious manner.

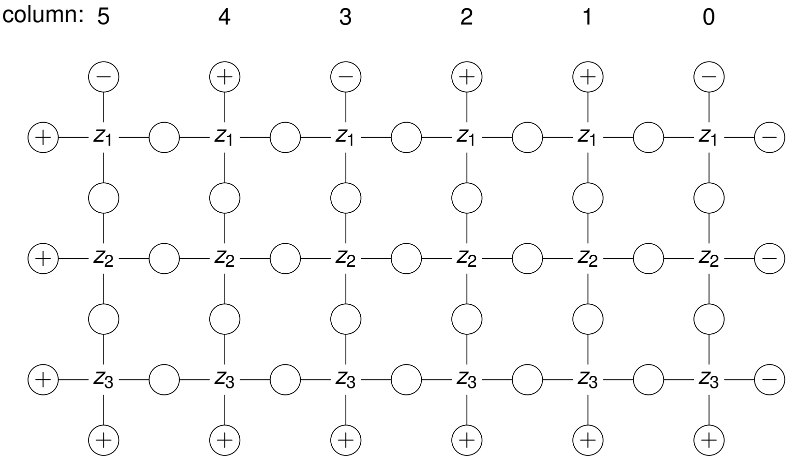
Fix Langlands parameters $z_1, \cdots, z_r$ and a corresponding representation of $\widetilde{GL}(r, F)$. Let λ be a partition. Let $p \in F$ be prime and

$$p^\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_r} \end{pmatrix}$$

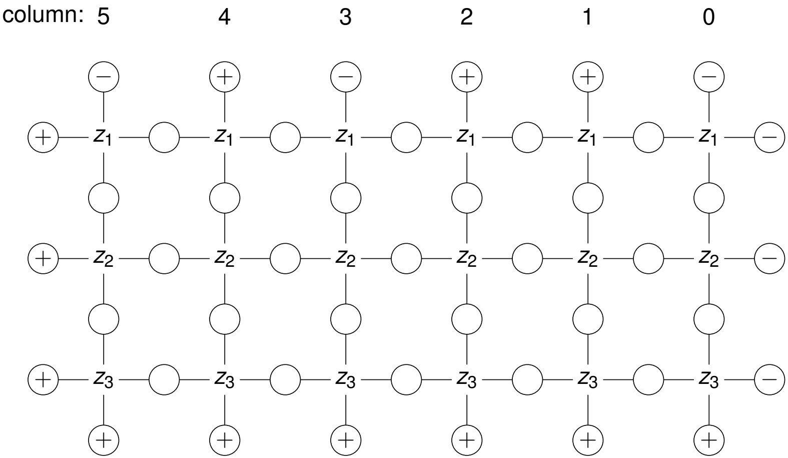
embedded in $\widetilde{GL}(r, F)$ by a standard section. We will describe a formula for the value $W(p^\lambda)$, one particular spherical Whittaker function for the representation index by z_i.

- We begin by recalling the ice-type models that were the second solvable lattice model to be studied.
- After Onsager solved the Ising model, Lieb, Sutherland and Baxter solved the field-free six-vertex model.
- In this context, Baxter employed the Yang-Baxter equation in an ingenious manner.
- The model we want is a variant of the six-vertex model.

Begin with a grid, usually (but not always) rectangular:

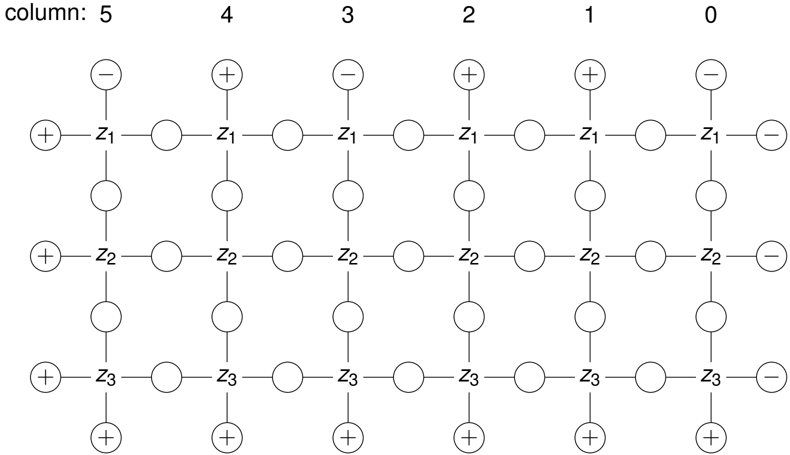


Edges are labeled with “spins” $\pm$. Boundary spins are fixed.



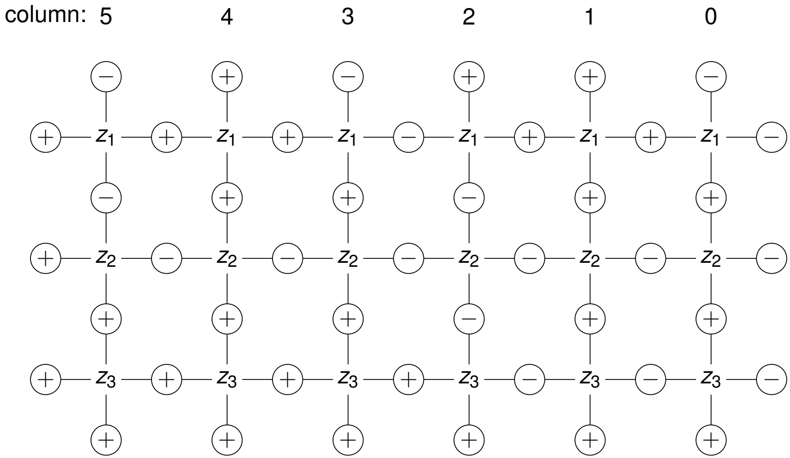
The boundary spins encode the partition λ.

Let $\rho = (r-1, r-2, \cdots, 2, 1, 0)$.



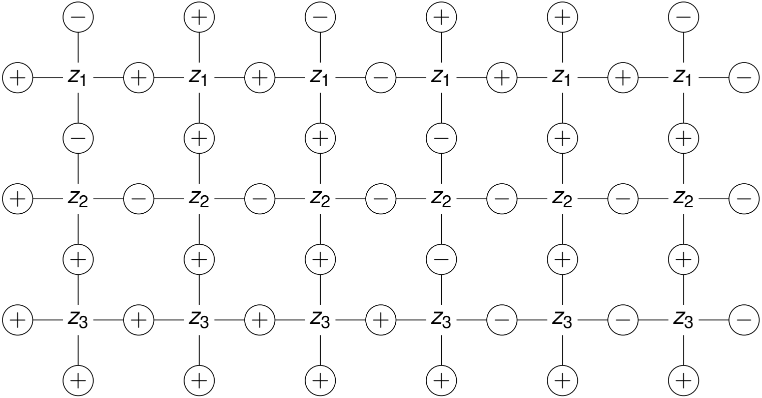
Put $-$ in the columns numbered by entries of $\lambda + \rho$.

A state of the system assigns spins to the interior edges:



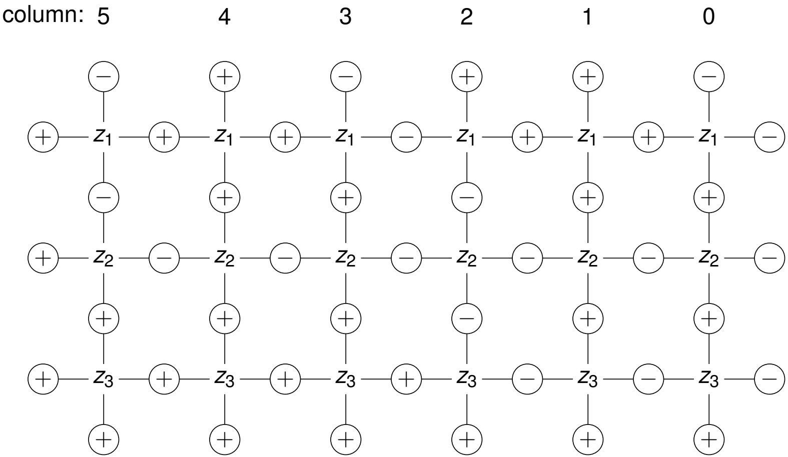
A state of the system assigns spins to the interior edges:

column: 5 4 3 2 1 0



Each state is assigned a Boltzmann weight.

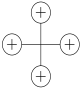
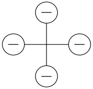
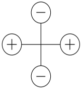
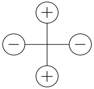
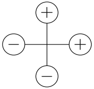
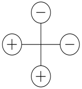
A state of the system assigns spins to the interior edges:



Each state is assigned a Boltzmann weight. The partition function Z is the sum over states.

Boltzmann Weights

The Boltzmann weight of a state is the product of Boltzmann weights, one for each vertex. These are given by the following table. Here $v = q^{-1}$ with q the residue cardinality. $g(a)$ is a Gauss sum.
$g(a)\, g(-a) = 1/v$ if $n \nmid a$,
$g(a) = -v$ if $n|a$ and $h(a) = (1 - v)$ if $n|a$ (0 otherwise).

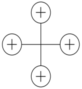
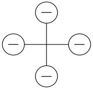
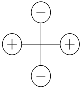
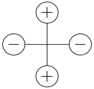
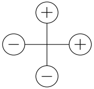
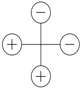
a_1	a_2	b_1	b_2	c_1	c_2
					
1	z_i	$g(a)$	z_i	$h(a)z_i$	1

Boltzmann Weights

The Boltzmann weight of a state is the product of Boltzmann weights, one for each vertex. These are given by the following table. Here $v = q^{-1}$ with q the residue cardinality. $g(a)$ is a Gauss sum.

$g(a)\, g(-a) = 1/v$ if $n \nmid a$,

$g(a) = -v$ if $n|a$ and $h(a) = (1 - v)$ if $n|a$ (0 otherwise).

a_1	a_2	b_1	b_2	c_1	c_2
					
1	z_i	$g(a)$	z_i	$h(a) z_i$	1

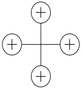
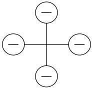
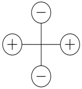
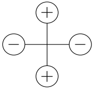
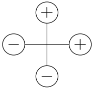
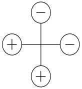
But what is a?

Boltzmann Weights

The Boltzmann weight of a state is the product of Boltzmann weights, one for each vertex. These are given by the following table. Here $v = q^{-1}$ with q the residue cardinality. $g(a)$ is a Gauss sum.

$g(a)\, g(-a) = 1/v$ if $n \nmid a$,

$g(a) = -v$ if $n|a$ and $h(a) = (1 - v)$ if $n|a$ (0 otherwise).

a_1	a_2	b_1	b_2	c_1	c_2
					
1	z_i	$g(a)$	z_i	$h(a)z_i$	1

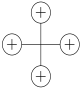
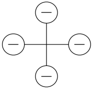
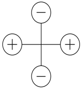
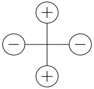
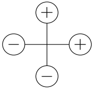
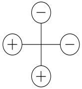
a is the voltage, the number of $+$ to the right of the vertex.

Boltzmann Weights

The Boltzmann weight of a state is the product of Boltzmann weights, one for each vertex. These are given by the following table. Here $v = q^{-1}$ with q the residue cardinality. $g(a)$ is a Gauss sum.

$g(a)\,g(-a) = 1/v$ if $n \nmid a$,

$g(a) = -v$ if $n|a$ and $h(a) = (1-v)$ if $n|a$ (0 otherwise).

a_1	a_2	b_1	b_2	c_1	c_2
					
1	z_i	$g(a)$	z_i	$h(a)z_i$	1

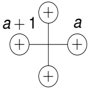
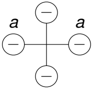
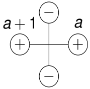
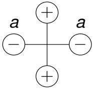
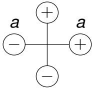
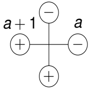
a is the voltage, the number of $+$ to the right of the vertex. This is not a purely local statistic, but it can be made local by incorporating it into the spins.

Boltzmann Weights

The Boltzmann weight of a state is the product of Boltzmann weights, one for each vertex. These are given by the following table. Here $v = q^{-1}$ with q the residue cardinality. $g(a)$ is a Gauss sum.

$g(a)\, g(-a) = 1/v$ if $n \nmid a$,

$g(a) = -v$ if $n|a$ and $h(a) = (1 - v)$ if $n|a$ (0 otherwise).

a_1	a_2	b_1	b_2	c_1	c_2
					
1	z_i	$g(a)$	z_i	$h(a)z_i$	1

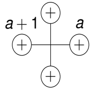
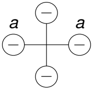
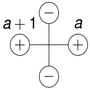
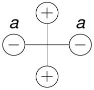
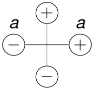
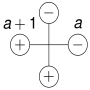
a is the voltage, the number of $+$ to the right of the vertex. This is not a purely local statistic, but it can be made local by incorporating it into the spins.

“Super” Edge Spins

- With the voltage taken into account, the possible “decorated” spins for horizontal edges are $+$ (a modulo n) and $-$ (only $a \equiv 0$ modulo n).

“Super” Edge Spins

- With the voltage taken into account, the possible “decorated” spins for horizontal edges are $+$ (a modulo n) and $-$ (only $a \equiv 0$ modulo n).

a_1	a_2	b_1	b_2	c_1	c_2
					
1	z_i	$g(a)$	z_i	$h(a)z_i$	1

- So when $-$ appears with $a \not\equiv 0$ mod n, we interpret that Boltzmann weight as zero.
- Denote this system as $\mathfrak{S}_\lambda$.

“Super” Edge Spins

- With the voltage taken into account, the possible “decorated” spins for horizontal edges are $+$ (a modulo n) and $-$ (only $a \equiv 0$ modulo n).
- BBCFG allowed $-$ spins with $n \nmid a$.

“Super” Edge Spins

- With the voltage taken into account, the possible “decorated” spins for horizontal edges are $+$ (a modulo n) and $-$ (only $a \equiv 0$ modulo n).
- BBCFG allowed $-$ spins with $n \nmid a$.
- For purely combinatorial reasons this change does not affect the partition function.

"Super" Edge Spins

- With the voltage taken into account, the possible "decorated" spins for horizontal edges are $+$ (a modulo n) and $-$ (only $a \equiv 0$ modulo n).
- BBCFG allowed $-$ spins with $n \nmid a$.
- For purely combinatorial reasons this change does not affect the partition function.
- This change from BBCFG makes possible the Yang-Baxter equation!

“Super” Edge Spins

- With the voltage taken into account, the possible “decorated” spins for horizontal edges are $+$ (a modulo n) and $-$ (only $a \equiv 0$ modulo n).
- BBCFG allowed $-$ spins with $n \nmid a$.
- For purely combinatorial reasons this change does not affect the partition function.
- This change from BBCFG makes possible the Yang-Baxter equation!
- The 1 negative and n positive possible edge spins can be thought of as basis vectors for a $\mathbb{Z}/2\mathbb{Z}$-graded “super” vector space that is a module for the Lie superalgebra $\mathfrak{gl}(1|n)$.

Partition Function

- If $n > 1$ the spherical Whittaker function is not unique.

Partition Function

- If $n > 1$ the spherical Whittaker function is not unique.
- We will give a formula for one function in the model.

Partition Function

- If $n > 1$ the spherical Whittaker function is not unique.
- We will give a formula for one function in the model.
- It is normalized so that

$$W(1) = \prod_{\alpha \in \Delta^+} (1 - v\mathbf{z}^{-n\alpha}).$$

Recall that $v = q^{-1}$ and $\mathbf{z} = \mathrm{diag}(z_1, \cdots, z_n)$ is an element of the Langlands dual group.

Partition Function

- If $n > 1$ the spherical Whittaker function is not unique.
- We will give a formula for one function in the model.
- It is normalized so that

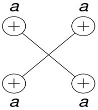
$$W(1) = \prod_{\alpha \in \Delta^+} (1 - v\mathbf{z}^{-n\alpha}).$$

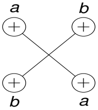
Recall that $v = q^{-1}$ and $\mathbf{z} = \mathrm{diag}(z_1, \cdots, z_n)$ is an element of the Langlands dual group.

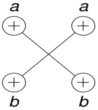
Theorem

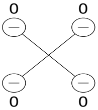
The partition function $Z(\mathfrak{S}_\lambda)$ equals $W(p^\lambda)$.

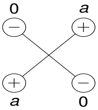
The R-matrix

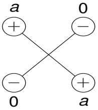
<table>
<tr>
<td>Another type of vertex,
the R-matrix R_{z_1,z_2}.
$1 \leqslant a \leqslant b \leqslant n$</td>
<td>

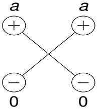
$z_2^n - v z_1^n$</td>
<td>

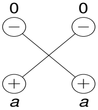
$g(a-b)(z_1^n - z_2^n)$</td>
</tr>
<tr>
<td>

$(1-v) z_1^{a-b} z_2^{n-a+b}$</td>
<td>

$z_1^n - v z_2^n$</td>
<td>

$v(z_1^n - z_2^n)$</td>
</tr>
<tr>
<td>

$z_1^n - z_2^n$</td>
<td>

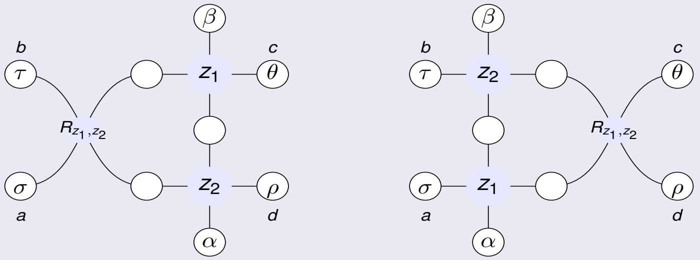
$(1-v) z_1^a z_2^{n-a}$</td>
<td>

$(1-v) z_1^{n-a} z_2^a$</td>
</tr>
</table>

The First Yang-Baxter Equation

Theorem

The following partition functions are equal.

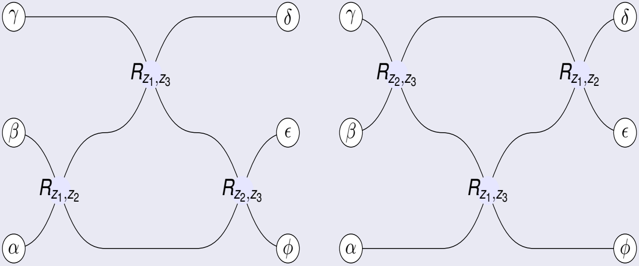


(As usual, the interior edge spins are summed.)

Another Yang-Baxter equation

Theorem

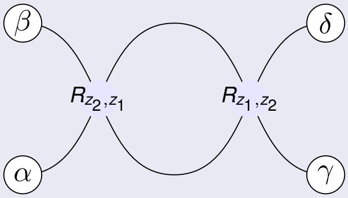
Fix z_1, z_2 and z_3 and (decorated) boundary spins. The following partition functions are equal:



Complementary Equation

Theorem

Let $\alpha, \beta, \gamma, \delta$ be decorated spins. Then the partition function



equals

$$\begin{cases} (z_1^n - vz_2^n)(z_2^n - vz_1^n) & \text{if } \alpha = \gamma,\ \beta = \delta \\ 0 & \text{otherwise.} \end{cases}$$

Digression: Lie superalgebras

- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.

Digression: Lie superalgebras

- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.
- A Lie superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Identities such as the Jacobi identity get sign changes.

Digression: Lie superalgebras

- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.
- A Lie superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Identities such as the Jacobi identity get sign changes.
- Given a $(m|n)$-dimensional super vector space, the Lie algebra of graded endomorphisms is called $\mathfrak{gl}(m|n)$.

Digression: Lie superalgebras

- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.
- A Lie superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Identities such as the Jacobi identity get sign changes.
- Given a $(m|n)$-dimensional super vector space, the Lie algebra of graded endomorphisms is called $\mathfrak{gl}(m|n)$.
- The affinization $\widehat{\mathfrak{gl}}(m|n)$ is an enlargement that has one $(m|n)$-dimensional module $V(z)$ for every $z \in \mathbb{C}^\times$.

Digression: Lie superalgebras

- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.
- A Lie superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Identities such as the Jacobi identity get sign changes.
- Given a $(m|n)$-dimensional super vector space, the Lie algebra of graded endomorphisms is called $\mathfrak{gl}(m|n)$.
- The affinization $\widehat{\mathfrak{gl}}(m|n)$ is an enlargement that has one $(m|n)$-dimensional module $V(z)$ for every $z \in \mathbb{C}^\times$.
- The enveloping algebra $U_v(\widehat{\mathfrak{gl}}(m|n))$ is an associative algebra with the same modules.

Digression: Lie superalgebras

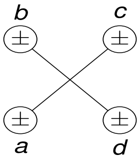
- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.
- A Lie superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Identities such as the Jacobi identity get sign changes.
- Given a $(m|n)$-dimensional super vector space, the Lie algebra of graded endomorphisms is called $\mathfrak{gl}(m|n)$.
- The affinization $\widehat{\mathfrak{gl}}(m|n)$ is an enlargement that has one $(m|n)$-dimensional module $V(z)$ for every $z \in \mathbb{C}^\times$.
- The enveloping algebra $U_v(\widehat{\mathfrak{gl}}(m|n))$ is an associative algebra with the same modules.
- The quantized enveloping algebra $U_v(\widehat{\mathfrak{gl}}(m|n))$ has same modules but $V(z) \otimes V(w) \to V(w) \otimes V(z)$ is modifed.

Digression: Lie superalgebras

- A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded one. If the odd/even dimensions are m, n graded dimension is $m|n$.
- A Lie superalgebra is a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. Identities such as the Jacobi identity get sign changes.
- Given a $(m|n)$-dimensional super vector space, the Lie algebra of graded endomorphisms is called $\mathfrak{gl}(m|n)$.
- The affinization $\widehat{\mathfrak{gl}}(m|n)$ is an enlargement that has one $(m|n)$-dimensional module $V(z)$ for every $z \in \mathbb{C}^\times$.
- The enveloping algebra $U_v(\widehat{\mathfrak{gl}}(m|n))$ is an associative algebra with the same modules.
- The quantized enveloping algebra $U_v(\widehat{\mathfrak{gl}}(m|n))$ has same modules but $V(z) \otimes V(w) \to V(w) \otimes V(z)$ is modifed.
- The perturbing endomorphism of $V(z) \otimes V(w)$ is called the R-matrix.

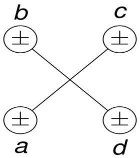
Tannakian Theory

Our goal is to interpret the Boltzmann weights as coefficients of a matrix $R_{z_1,z_2} \in \mathrm{End}(V(z_1) \otimes V(z_2))$ Where $V(z_i)$ are $(1|n)$-dimensional.



Tannakian Theory

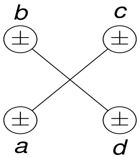
Our goal is to interpret the Boltzmann weights as coefficients of a matrix $R_{z_1,z_2} \in \mathrm{End}(V(z_1) \otimes V(z_2))$ Where $V(z_i)$ are $(1|n)$-dimensional.



- The even graded part of $V(z)$ is one-dimensional spanned by a vector labeled $-$ with voltage 0.

Tannakian Theory

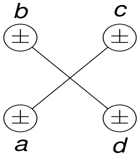
Our goal is to interpret the Boltzmann weights as coefficients of a matrix $R_{z_1,z_2} \in \mathrm{End}(V(z_1) \otimes V(z_2))$ Where $V(z_i)$ are $(1|n)$-dimensional.



- The even graded part of $V(z)$ is one-dimensional spanned by a vector labeled $-$ with voltage 0.
- The odd graded part is spanned by vectors labeled and $+$ (with voltage a mod n).

Tannakian Theory

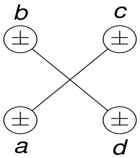
Our goal is to interpret the Boltzmann weights as coefficients of a matrix $R_{z_1,z_2} \in \mathrm{End}(V(z_1) \otimes V(z_2))$ Where $V(z_i)$ are $(1|n)$-dimensional.



- The even graded part of $V(z)$ is one-dimensional spanned by a vector labeled $-$ with voltage 0.
- The odd graded part is spanned by vectors labeled and $+$ (with voltage a mod n).
- Let $\tau(x \otimes y) = y \otimes x$. Interpret $\tau R_{z_1,z_2}$ as an intertwining map $V(z_1) \otimes V(z_2) \to V(z_2) \otimes V(z_1)$.

Tannakian Theory

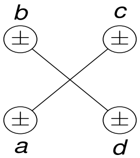
Our goal is to interpret the Boltzmann weights as coefficients of a matrix $R_{z_1,z_2} \in \mathrm{End}(V(z_1) \otimes V(z_2))$ Where $V(z_i)$ are $(1|n)$-dimensional.



- The even graded part of $V(z)$ is one-dimensional spanned by a vector labeled $-$ with voltage 0.
- The odd graded part is spanned by vectors labeled and $+$ (with voltage a mod n).
- Let $\tau(x \otimes y) = y \otimes x$. Interpret $\tau R_{z_1,z_2}$ as an intertwining map $V(z_1) \otimes V(z_2) \to V(z_2) \otimes V(z_1)$.
- Yang-Baxter equation means the category generated by the $V(z)$ is a braided monoidal category.

Tannakian Theory

Our goal is to interpret the Boltzmann weights as coefficients of a matrix $R_{z_1,z_2} \in \mathrm{End}(V(z_1) \otimes V(z_2))$ Where $V(z_i)$ are $(1|n)$-dimensional.



- The even graded part of $V(z)$ is one-dimensional spanned by a vector labeled $-$ with voltage 0.
- The odd graded part is spanned by vectors labeled and $+$ (with voltage a mod n).
- Let $\tau(x \otimes y) = y \otimes x$. Interpret $\tau R_{z_1,z_2}$ as an intertwining map $V(z_1) \otimes V(z_2) \to V(z_2) \otimes V(z_1)$.
- Yang-Baxter equation means the category generated by the $V(z)$ is a braided monoidal category.
- Tannakian theory (Saavedra-Rivano, Ulbrich, Majid): interpret this category as modules over a quantum group.

Quantum Group

- The affinization $\widehat{\mathfrak{gl}}(1|n)$ has one $(1|n)$-dimensional module for each $z \in \mathbb{C}^\times$.

Quantum Group

- The affinization $\widehat{\mathfrak{gl}}(1|n)$ has one $(1|n)$-dimensional module for each $z \in \mathbb{C}^\times$.
- Looks good ...

Quantum Group

- The affinization $\widehat{\mathfrak{gl}}(1|n)$ has one $(1|n)$-dimensional module for each $z \in \mathbb{C}^\times$.
- Looks good ... but its R-matrix does not involve Gauss sums.

Quantum Group

- The affinization $\widehat{\mathfrak{gl}}(1|n)$ has one $(1|n)$-dimensional module for each $z \in \mathbb{C}^\times$.
- Looks good ... but its R-matrix does not involve Gauss sums.
- Drinfeld twisting is a procedure that modifies a Hopf algebra.

Quantum Group

- The affinization $\widehat{\mathfrak{gl}}(1|n)$ has one $(1|n)$-dimensional module for each $z \in \mathbb{C}^\times$.
- Looks good ... but its R-matrix does not involve Gauss sums.
- Drinfeld twisting is a procedure that modifies a Hopf algebra.
- This procedure can be used to put Gauss sums into the R-matrix.

Quantum Group

- The affinization $\widehat{\mathfrak{gl}}(1|n)$ has one $(1|n)$-dimensional module for each $z \in \mathbb{C}^\times$.
- Looks good ... but its R-matrix does not involve Gauss sums.
- Drinfeld twisting is a procedure that modifies a Hopf algebra.
- This procedure can be used to put Gauss sums into the R-matrix.

Theorem

R_{z_1,z_2} is the R-matrix of a Drinfeld twist of $U_v(\mathfrak{gl}(1|n))$.

The Scattering Matrix

- Let $\pi_{\mathbf{z}}$ be the spherical principal series representation indexed by $\mathbf{z}$.

The Scattering Matrix

- Let $\pi_{\mathbf{z}}$ be the spherical principal series representation indexed by $\mathbf{z}$.
- The dimension of the space of spherical Whittaker functions is n^r.

The Scattering Matrix

- Let $\pi_{\mathbf{z}}$ be the spherical principal series representation indexed by $\mathbf{z}$.
- The dimension of the space of spherical Whittaker functions is n^r.
- If $w \in W$, the Weyl group, then there is an intertwining operator $\pi_{\mathbf{z}} \to \pi_{w\mathbf{z}}$, and so there is a scattering matrix on the Whittaker functions.

The Scattering Matrix

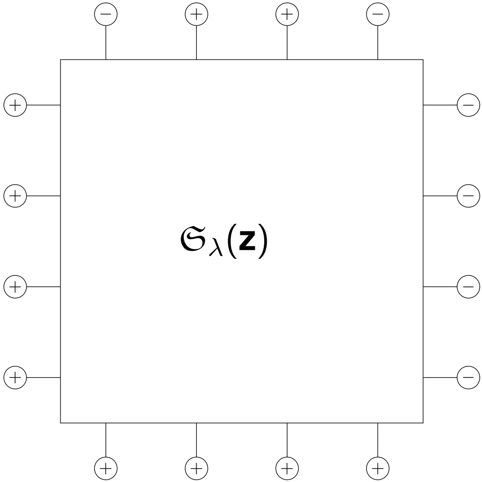
- Let $\pi_{\mathbf{z}}$ be the spherical principal series representation indexed by $\mathbf{z}$.
- The dimension of the space of spherical Whittaker functions is n^r.
- If $w \in W$, the Weyl group, then there is an intertwining operator $\pi_{\mathbf{z}} \to \pi_{w\mathbf{z}}$, and so there is a scattering matrix on the Whittaker functions.
- This was calculated by Kazhdan and Patterson. It reappears in work of Chinta–Gunnells, Chinta-Offen and McNamara.

The Scattering Matrix

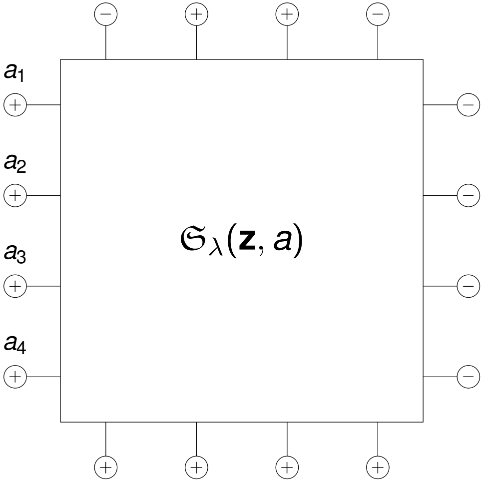
- Let $\pi_{\mathbf{z}}$ be the spherical principal series representation indexed by $\mathbf{z}$.
- The dimension of the space of spherical Whittaker functions is n^r.
- If $w \in W$, the Weyl group, then there is an intertwining operator $\pi_{\mathbf{z}} \to \pi_{w\mathbf{z}}$, and so there is a scattering matrix on the Whittaker functions.
- This was calculated by Kazhdan and Patterson. It reappears in work of Chinta–Gunnells, Chinta-Offen and McNamara.
- The scattering matrix is somewhat complicated and involves Gauss sums.

Can we model this using $R_{z_i,z_{i+1}}$ when $w = s_i$?

We described one Whittaker function as a partition function.

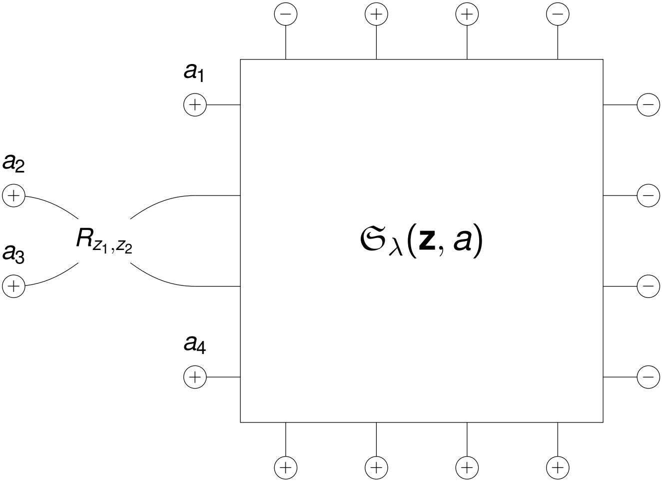


We now describe all Whittaker functions as partition function.



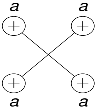
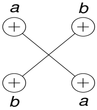
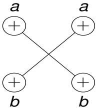
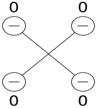
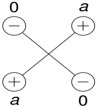
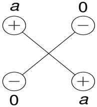
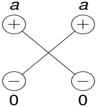
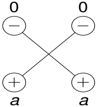
Decompose W by imposing left edge voltages.

$R_{z_i,z_{i+1}}$ gives the scattering matrix of the intertwining operator:



Attach the R-matrix and use Yang-Baxter equation.

Intertwining operators as R-matrices

<table>
<tr>
<td>We attached $R_{z_i,z_{i+1}}$ where it only sees $+$ edges.</td>
<td>
$z_2^n - v z_1^n$</td>
<td>
$g(a-b)(z_1^n - z_2^n)$</td>
</tr>
<tr>
<td>
$(1-v)z_1^{a-b}z_2^{n-a+b}$</td>
<td>
$z_1^n - v z_2^n$</td>
<td>
$v(z_1^n - z_2^n)$</td>
</tr>
<tr>
<td>
$z_1^n - z_2^n$</td>
<td>
$(1-v)z_1^a z_2^{n-a}$</td>
<td>
$(1-v)z_1^{n-a}z_2^a$</td>
</tr>
</table>

Intertwining operators as R-matrices

So only the $+$ part of the R-matrix is involved in the scattering matrix of the intertwining operators on the Whittaker model. Let $V_+(z)$ be the n-dimensional odd graded subspace.

Theorem

There exists an isomorphism $\theta_{\mathbf{z}}$ of the space of Whittaker functionals with $V_+(z_1) \otimes \cdots V_+(z_r)$ such that:

$$\begin{array}{ccc}
\mathcal{W}_{\mathbf{z}} & \xrightarrow{\theta_{\mathbf{z}}} & V_+(z_1) \otimes \cdots \otimes V_+(z_i) \otimes V_+(z_{i+1}) \otimes \cdots \otimes V(z_r) \\
\downarrow \mathcal{A}^*_{s_i} & & \downarrow I_{V_+(z_1)} \otimes \cdots \otimes \tau R^+_{z_i, z_{i+1}} \otimes \cdots \otimes I_{V_+(z_r)} \\
\mathcal{W}_{\mathbf{s_i z}} & \xrightarrow{\theta_{\mathbf{s_i z}}} & V_+(z_1) \otimes \cdots \otimes V_+(z_{i+1}) \otimes V_+(z_i) \otimes \cdots \otimes V(z_r)
\end{array}$$

commutes

Here R^+ is the R-matrix for a Drinfeld twist of $U_v(\widehat{\mathfrak{gl}}(n))$.

Summary

- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(n))$ precisely models the intertwining operators for the n-fold cover of $\mathrm{GL}(r)$.

Summary

- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(n))$ precisely models the intertwining operators for the n-fold cover of $\mathrm{GL}(r)$.
- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(1|n))$ is involved in the internal states of the system whose partition function represents the Whittaker function but it does not appear in the external manifestation of the Whittaker function.

Summary

- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(n))$ precisely models the intertwining operators for the n-fold cover of $\mathrm{GL}(r)$.
- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(1|n))$ is involved in the internal states of the system whose partition function represents the Whittaker function but it does not appear in the external manifestation of the Whittaker function.
- Hence it is unclear whether the superalgebra has a deeper significance in this theory.

Summary

- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(n))$ precisely models the intertwining operators for the n-fold cover of $\mathrm{GL}(r)$.
- the R-matrix of $U_v(\widehat{\mathfrak{gl}}(1|n))$ is involved in the internal states of the system whose partition function represents the Whittaker function but it does not appear in the external manifestation of the Whittaker function.
- Hence it is unclear whether the superalgebra has a deeper significance in this theory.
- The space of Whittaker models is isomorphic to $V_+(z_1) \otimes \cdots \otimes V_+(z_r)$, where $V_+(z)$ is the ungraded n-dimensional standard module of $U_v(\widehat{\mathfrak{gl}}(n))$.