Fourier Coefficients for Theta Representations on Metaplectic Groups

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- Work with global setup
- Also describe the local picture
- Work with covers of general linear groups
- Theta Representations = residues of Eisenstein series
- Generalizations of Whittaker coefficients

Metaplectic Groups

- F: number field such that $\mu_n \subset F$
- $\mathbb{A}$: adele ring of F
- $(B, T, U) =$ (Borel, torus, unipotent) in GL_r
- Fix an n-fold cover of $\mathrm{GL}_r(\mathbb{A})$

$$1 \to \mu_n \to \widetilde{\mathrm{GL}_r}(\mathbb{A}) \xrightarrow{p} \mathrm{GL}_r(\mathbb{A}) \to 1$$

  (central extension).
- Fix $\epsilon : \mu_n \to \mathbb{C}^\times$
- Only consider representations of $\widetilde{\mathrm{GL}_r}(\mathbb{A})$ such that μ_n acts via ϵ.
- For subgroup $H \subset \mathrm{GL}_r(\mathbb{A})$, define $\widetilde{H} = p^{-1}(H)$.

Eisenstein Series

- Local: Induced Representations
- $\widetilde{T}$ is not abelian, but two-step nilpotent
- Representation theory of $\widetilde{T}$: Stone-von Neumann theorem
- Starting with $\chi : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, define

$$\chi||^{s_1} \otimes \cdots \otimes \chi||^{s_r} : T(\mathbb{A}) \to \mathbb{C}^\times.$$

This determines (non-uniquely)

$$\begin{aligned} &\text{character of } Z(\widetilde{T}(\mathbb{A})) \\ \xrightarrow{\text{Induction}} &\text{irr rep of } \widetilde{T}(\mathbb{A}) \\ \longrightarrow &\text{irr rep of } \widetilde{B}(\mathbb{A}) \\ \xrightarrow{\text{Induction}} &\text{rep of } \widetilde{\mathrm{GL}}_r(\mathbb{A}) \\ \xrightarrow{\text{Averaging}} &\text{Eisenstein series } E(g, \underline{s}, f) \end{aligned}$$

Theta Representations

- The Eisenstein series $E(g, \underline{s}, f)$ has a pole at $\underline{s} = \underline{s}_0$.
- Define (Theta function)

$$\theta(f, g) = \mathrm{Res}_{\underline{s}=\underline{s}_0} E(g, \underline{s}, f).$$

- These functions form an irreducible square integrable automorphic representation of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$.
- $\rightsquigarrow \Theta_r$, theta representation.

- Local Theory: $\Theta_r \cong \otimes'_v \Theta_{r,v}$ (restricted tensor product).
- Locally, $\Theta_{r,v}$ is the irreducible quotient of a reducible principal series representation.
- When $n = 1$, Θ_r is one-dimensional.
- When $n = r = 2$, this recovers the Jacobi theta function.

Whittaker Models of Theta Representations

Theorem (Kazhdan-Patterson, 1984)

(Both locally and globally.)

1. *When* $n \geq r$, Θ_r *is generic.*
2. *When* $n < r$, Θ_r *is not generic.*
3. *When* $n = r$ *or* $n = r + 1$ *(in this case, only for certain covers), uniqueness of Whittaker models holds for* Θ_r.

Remarks

- When $n = r$, the Whittaker coefficients are expressed in terms of nth order Gauss sum.
- For higher covers of SL_2, new information can be obtained by using decent method (Friedberg-Ginzburg).

Applications: Rankin-Selberg integrals for symmetric power L-functions

- Symmetric square for $\mathrm{GL}(r)$:
  Bump-Ginzburg 1992
  generalizing works of Shimura 1975, Gelbert-Jacquet 1978, Patterson-Piatetski-Shapiro 1989
- Twisted symmetric square for $\mathrm{GL}(r)$: Takeda 2014
- Symmetric cube for $\mathrm{GL}(2)$: Bump-Ginzburg-Hoffstein 1996.

Questions

Suppose $n < r$. Then Θ_r is not generic.

- Other types of Fourier coefficients?
- Unique model?
- Applications?

Fourier Coefficients

- f: automorphic form
- V: unipotent subgroup
- ψ_V: character on V
- Globally, define Fourier coefficient

$$(V, \psi_V) \rightsquigarrow \int_{V(F)\backslash V(\mathbb{A})} f(vg)\psi_V(v)\, dv.$$

- Locally, define model/functional as an element in

$$\mathrm{Hom}_V(\pi, \psi_V) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_V^G \psi_V).$$

Semi-Whittaker Coefficients

- $\lambda = (r_1 \cdots r_k)$: partition of r
  $\rightsquigarrow (U, \psi_\lambda)$
- $(P_\lambda, M_\lambda, U_\lambda) =$(parabolic, Levi, unipotent radical)
- $M_\lambda = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k} \hookrightarrow \mathrm{GL}_r$
- $\psi : F \backslash \mathbb{A} \to \mathbb{C}^\times$: additive character
- $\psi_\lambda : U(F) \backslash U(\mathbb{A}) \to \mathbb{C}^\times$
  - acts as ψ on the simple root subgroups contained in M_λ;
  - trivially otherwise.

For example, if the partition is $(3^2 2)$, then

$$u = \left(\begin{array}{ccc|ccc|cc} 1 & a & * & * & * & * & * & * \\ & 1 & b & * & * & * & * & * \\ & & 1 & c & * & * & * & * \\ \hline & & & 1 & d & * & * & * \\ & & & & 1 & e & * & * \\ & & & & & 1 & f & * \\ \hline & & & & & & 1 & g \\ & & & & & & & 1 \end{array}\right)$$

$$\psi_\lambda(u) = \psi(a + b + d + e + g).$$

Definition (λ-semi-Whittaker coefficient)

Given $\theta \in \Theta_r$. The λ-semi-Whittaker coefficient of θ is

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du.$$

Remark

When $n = 2$ and $\lambda = (2^k)$ or (2^k1) (depending on the parity of r), these coefficients were used in Bump-Ginzburg's work on symmetric square L-functions for $\mathrm{GL}(r)$.

Theorem (C, 2016)

1. *If $r_i > n$ for some i, then*

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du = 0$$

   for all choices of data.

2. *If $r_i \leq n$ for all i, then*

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\psi_\lambda(u)\, du \neq 0$$

   for some choice of data.

3. *Let v be a finite place. If $r = mn$, and $\lambda = (n^m)$, then*

$$\dim \mathrm{Hom}_{U(F_v)}(\Theta_{r,v}, \psi_{\lambda,v}) = 1.$$

Remark 1

For parts (1) and (2), local versions are also true (expressed in terms of twisted Jacquet modules).

Remark 2

Induction in stages statement:

- Constant terms of Θ_r along $U_\lambda = \Theta_{M_\lambda}$ on $\widetilde{M_\lambda}$.
- Θ_{M_λ} "$\cong$" $\Theta_{r_1} \otimes \cdots \otimes \Theta_{r_k}$.
  The tensor product on the right-hand side is the metaplectic tensor product.
  (Local version: Kable and Mezo; global version: Takeda).

Remark 3

Explicit formula:

$$\dim J_{U,\psi_\lambda}(\Theta_r) = (\text{correction factor}) \times \prod_{i=1}^{k} \dim J_{U_{\mathrm{GL}_{r_i}},\psi_{Wh}}(\Theta_{r_i}).$$

Fourier Coefficients Associated with Unipotent Orbits

- Unipotent orbits $\leftrightsquigarrow$ Jordan decomposition $\leftrightsquigarrow$ partitions of r
- Unipotent orbit $\mathcal{O} \rightsquigarrow (U_2(\mathcal{O}), \psi_{U_2(\mathcal{O})})$.
- The maximal unipotent orbit $(r) \rightsquigarrow$ Whittaker coefficients.
- Partial order on the set of unipotent orbits: $\mathcal{O}_1 = (p_1 \cdots p_k)$ and $\mathcal{O}_2 = (q_1 \cdots q_l)$. Then

$$\mathcal{O}_1 \geq \mathcal{O}_2 \Leftrightarrow p_1 + \cdots + p_i \geq q_1 + \cdots + q_i \text{ for all } i.$$

Example: $r = mn$ and $\mathcal{O} = (n^m)$

$$\left\{ \begin{pmatrix} I_m & X_1 & * & * & \cdots & * \\ & I_m & X_2 & * & \cdots & * \\ & & I_m & X_3 & \cdots & * \\ & & & \ddots & \ddots & * \\ & & & & I_m & X_{n-1} \\ & & & & & I_m \end{pmatrix} : X_j \in \mathrm{Mat}_{m\times m} \right\}.$$

$$\psi_{U_2(\mathcal{O})}(u) = \psi(\mathrm{tr}(X_1 + \cdots + X_{n-1})).$$

Let $\lambda = \mathcal{O} = (3^2)$. Then

$$\begin{pmatrix} 1 & a & * & * & * & * \\ & 1 & b & * & * & * \\ & & 1 & c & * & * \\ & & & 1 & d & * \\ & & & & 1 & e \\ & & & & & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & & a & * & * & * \\ & 1 & * & b & * & * \\ & & 1 & & c & * \\ & & & 1 & * & d \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$\psi_\lambda(u) = \psi(a+b+d+e), \qquad \psi_{U_2(\mathcal{O})} = \psi(a+b+c+d).$$

- Unipotent orbit $\mathcal{O}$ attached to an automorphic representation π:

  $\mathcal{O}$ supports some nonzero Fourier coefficient;

  $\mathcal{O}'$ larger than or incomparable with $\mathcal{O}$ does not support any Fourier coefficient.
- The question of determining the maximal unipotent orbit that supports a nonzero coefficient is often important (e.g. descent methods)

Theorem (C,2016)

1. *Write* $r = an + b$ *such that* $a \in \mathbb{Z}_{\geq 0}$ *and* $0 \leq b < n$*. Then both locally and globally* $\mathcal{O}(\Theta_r) = (n^a b)$.
2. *Let* v *be a finite place such that* $|n|_v = 1$ *and* $\Theta_{r,v}$ *is unramified. If* $r = mn$ *and* $\mathcal{O} = (n^m)$*, then*

$$\dim \mathrm{Hom}_{U_2(\mathcal{O})(F_v)}(\Theta_{r,v}, \psi_{U_2(\mathcal{O}),v}) = 1.$$

Applications

Doubling Constructions for Covering Groups

Joint work with Friedberg, Ginzburg, and Kaplan.

Copies of theta representations are used to construct Eisenstein series on covers of the classical groups.

Generating Functions on Covering Groups

Work by Ginzburg (confirming a conjecture of Bump and Friedberg).
This is a new-way integral.

The unique functionals on theta representations are used to compute generating functions on covering groups.

Degenerate Eisenstein Series

- Fix a partition $\mu = (p_1 \cdots p_m)$ of r
- Consider degenerate Eisenstein series corresponding to $\mathrm{Ind}_{P_\mu}^{\mathrm{GL}_r} \delta_{P_\mu}^{s}$.
- $\mu^\top = (q_1 \cdots q_n)$: transpose of μ

Theorem (Rough version)

For semi-Whittaker coefficients and Fourier coefficients associated with unipotent orbits, the orbit $\mu^\top$ is the maximal partition that supports nonzero coefficients.

- This confirms a conjecture of David Ginzburg.
- Difficulty: vanishing for incomparable orbits.
- Idea: express Weyl group elements in a certain way, and apply double coset computation.