Diary on a map

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Introduction

- map: $(x', y') = F(x, y)$ diffeomorphism
- integrable: $G(x,y) = G(x',y')$, G smooth, G not constant, $G^{-1}(g)$ bounded
- example: $\theta' = \theta + 2\pi r(I)$, $I' = I$; $G(\theta, I) = I$, $(\theta, I) \in S^1 x I$

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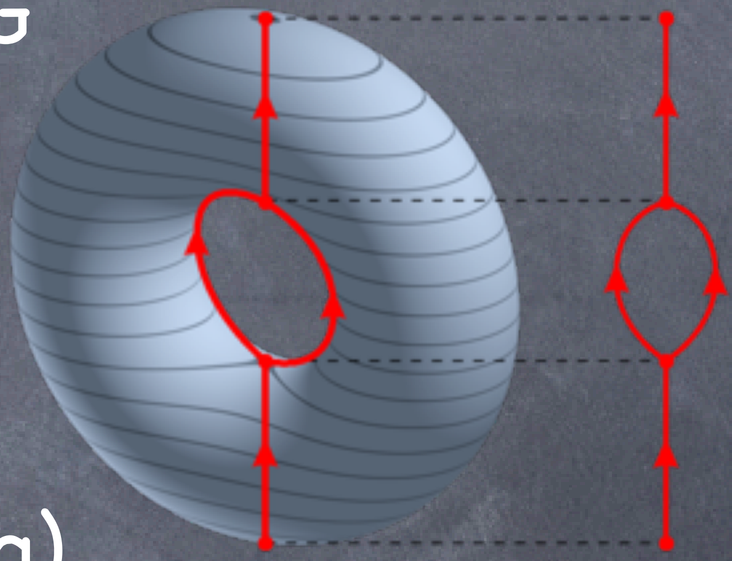
Reeb graph

- Reeb graph of a continuous function G on a topological space X describes change of level sets
- quotient X/~ where a~b if a and b belong to the same component of $G^{-1}(g)$ for some g
- each point in an edge: topological circle
- on each edge: rotation function r(g)



source: Wikipedia

Equivalence

- If there is only a single critical point in each level then two maps are topologically conjugate if their rotation functions are conjugate on each edge:
  $r_1(g) = r_2(s(g))$ for some homeomorphism s
- The general case has been treated by Bolsinov, Fomenko (1994), for flows by Izosimov, Khesin, Mousavi (2015) up to symplectic equivalence
- The rotation function is the main dynamical object of an integrable map

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How smooth is the integral G?

- Theorem (Taimanov, Bolsinov 2000): There exists an integrable geodesic flow with analytic metric which has positive topological entropy. One integral is C^∞ and cannot be made smoother.

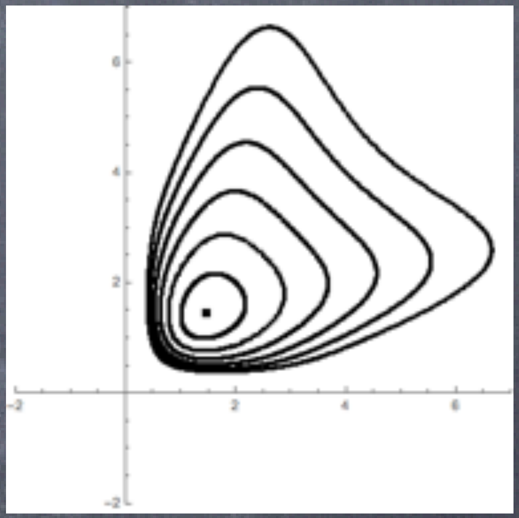
$$\exp(-(uv)^{-2}) \sin(c \log |uv^{-1}|)$$

- For the quantisation of this system see HRD, Bolsinov, Veselov, CMP 2006
- Is there such an example for maps?

Prologue

- Viallet (2008) introduced a bi-rational map that appears integrable in the real, but has non-zero algebraic entropy:

$$x' = \frac{1}{y}\left(x + \frac{1}{x}\right),$$
$$y' = x$$



- This implies that the integral cannot be algebraic.
- Can we find this integral?

$$x' = (x + x^{-1})y^{-1}, y' = x$$

Day 1: Lagrangian Form

- F has invariant measure $\frac{\mathrm{d}x\,\mathrm{d}y}{xy}$
- F leaves the positive quadrant invariant
- New variables x = exp(u), y = exp(v)
- Gives area preserving map
  (u',v') = H(u,v) = (-v + log(2cosh(u)), u)
- with Lagrangian generating function
  L(u,u') = $(u-u')^2$ - V(u)
  where V'(u) = log(2 cosh u)
- Is it a standard like map, Suris (1989)? No!

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Day 2: Periodic Points

- $H = S \circ R$ is composition of involutions:
  $R(u,v) = (v,u)$, $S(u,v) = (-u + \log 2 \cosh v, v)$
  $R \circ R = Id$, $S \circ S = Id$

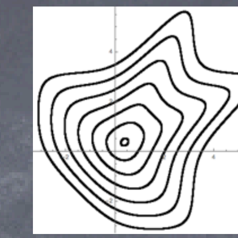
- Gives $RS = H^{-1}$, $RSRR = H^{-1}$, $RHR = H^{-1}$, $RH^nR = H^{-n}$

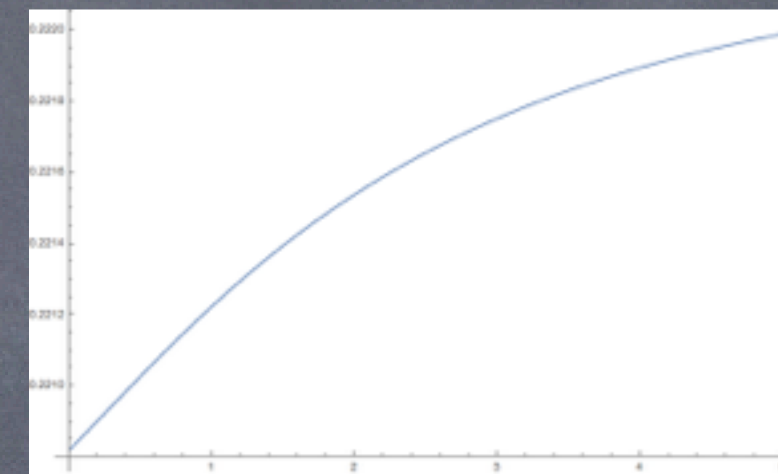
- Fix $R = \{ x = y \}$, Fix $R^n = H^n(\text{Fix } R)$

- Intersection of Fix R and Fix R^n is a periodic point of period 2n:
  $H^n(t,t) = (s,s)$, $RH^nR(t,t) = (s,s)$, $H^n(t,t) = H^{-n}(t,t)$

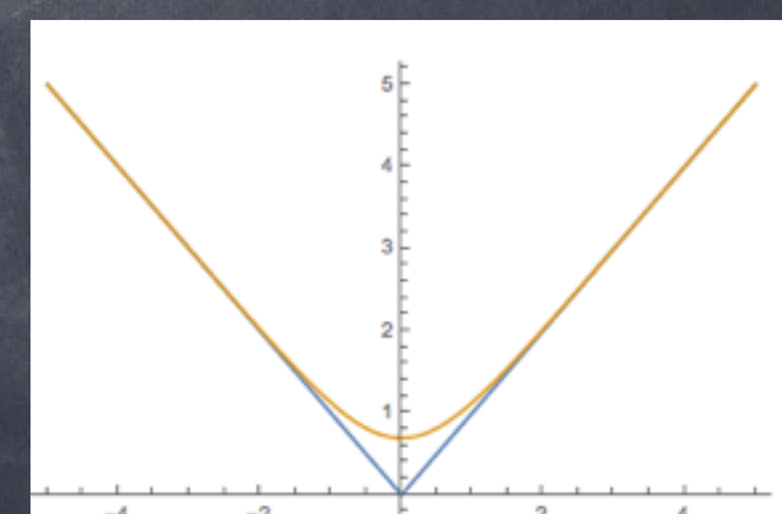
- 1-dimensional search, high precision, period 86, 95, 104,... with multipliers = 1 to 25 digits

Day 3: Rotation Function





- Single family of invariant circles, graph is a line segment
- The fixed point has multiplier $\exp(2\pi i r_0)$, $r_0 \approx 0.220818$
- Near infinity r approaches 2/9
- High precision numerics to compute rotation function using continued fraction expansion to accelerate convergence
- $V'(u) = \log 2\cosh u \approx |u|$ for large u
  $(u',v') = (-v + |u|), u)$ has period 9
- $19/86, 21/95, 23/104, \ldots \in [r_0, 2/9]$



status

- Numerical tests of Day 2 and 3 are convincing evidence that the map is in fact integrable
- For other maps, e.g. with $V'(u) = (1 + u^2)^{-1/2}$ whose phase portrait looks similar these tests fail:
    - there are many hyperbolic orbits, and
    - the rotation function is not smooth

Birkhoff Normal Form

- Remove non-linear terms near a fixed point by near-identity symplectic coordinate transformations
- Practically done using Lie series
- Theorem (Birkhoff): If the multiplier of the fixed point is not a root of unity (NR), then all non-linear terms but powers of action variables can be formally removed. Convergence ⇒ analytic integral.
- Analytically integrable & NR ⇒ convergence Ito 1989

  without NR: Zung 2005

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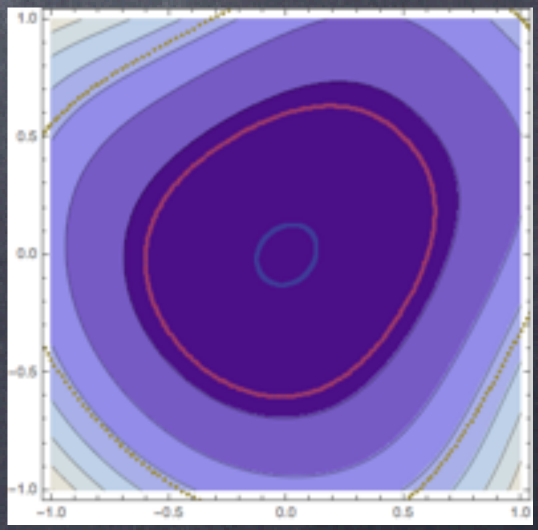
Day 4: Birkhoff Normal Form

- Shift coordinates so that $H(0,0) = (0,0)$:
  $(u',v') = H(-v + \log(\cosh(u) + \sinh(u) 2\cos 2\pi r_0), u)$

- BNF gives map in action-angle variables:
  $(\theta', I') = (\theta + 2\pi(r_0 + c_1 I + c_2 I^2 + \ldots), I)$ (ex. 1!)
  can find c_1, c_2, …, they are complicated:

$$c_1 = \frac{(t^2-3)(3t^2-1)(7t^2-5)}{16t^3(t^2+1)} \quad c_2 = \frac{(t^2-3)(3t^2-1)(833t^{12}-1202t^{10}-3089t^8+4932t^6+623t^4-2610t^2+705)}{1024(t-1)t^7(t+1)(t^2+1)^2}$$

$13t^6 + 11t^4 + 31t^2 - 31 = 0$ take real solution



- Radius of convergence?

- Which $f(I)$ would be the "best" one?

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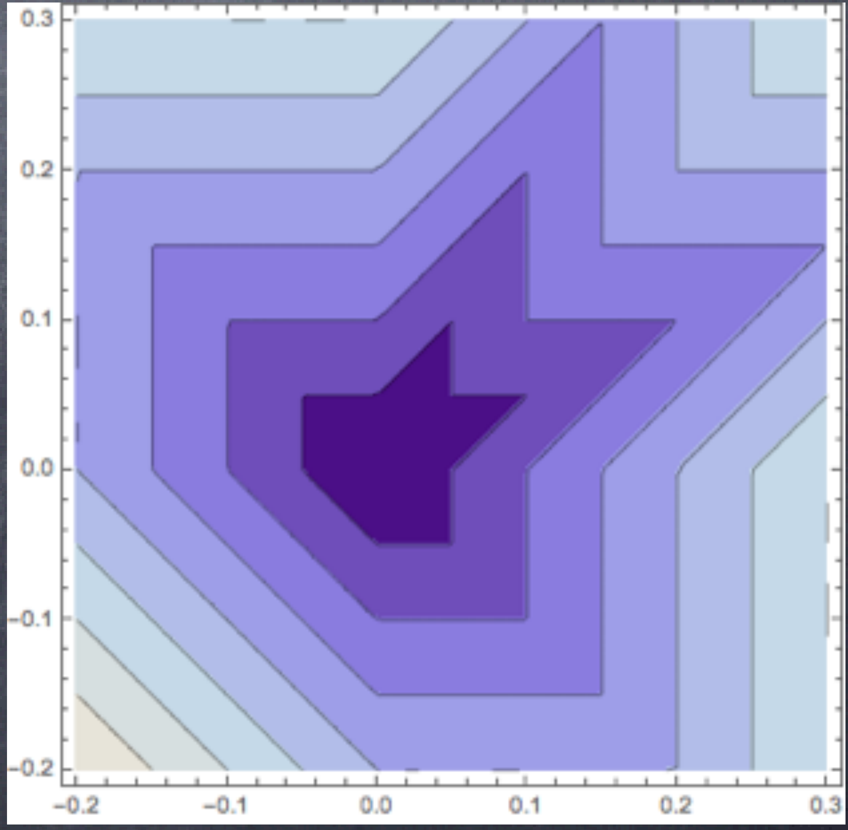
Day 5: Near Infinity

actual map:

- (u′,v′) = L(u,v) = (-v + |u|), u)
- L is periodic with period 9 (Knuth 1985)
- L is homogeneous in u,v: scaling!
- Construct integral for L by averaging:

$$G(u,v) = \sum_{i=-4}^{4} S(L^i(u,v)),$$

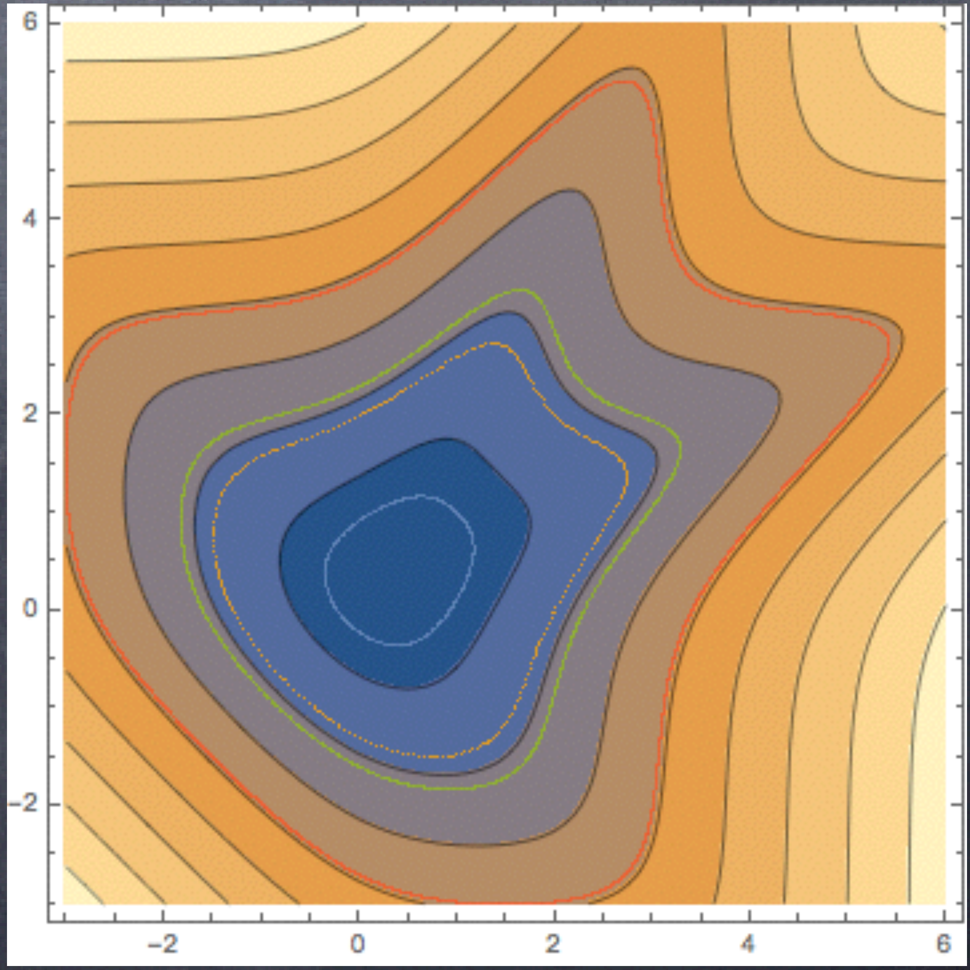
$$S(u,v) = u + v$$



Day 6: Approximate Integral

- Combine the local integral from Birkhoff normal form with the "global" integral from averaging
- Denote by I the truncated action map obtained from the Birkhoff normal form:

$$G(u,v) = \sum_{i=-4}^{4} I(H^i(u,v))$$



Day 7: Integrable?

- Day 6 integral is only an approximate integral, because BNF is only known to finite order, and may not converge globally
- So I went back to Day 2, periodic orbits: circles with rational rotation number are split into resonance zones
- r = 19/86: hyperbolic multiplier $\approx 1 \pm 1.4 * 10^{-29}$, elliptic multiplier $\approx \exp(\pm 2\pi\ 1.05 * 10^{-29})$,
- Similarly for other rational r: multipliers extremely close to 1, but not equal to 1

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Epilogue

- Does this prove non-integrability? No!
Graph could be a tree with many branches.
- Proving non-integrability based on a finite number of resonance zones would work if the integral is known to have only a finite number of critical points, but we do not have such a bound.
- Is there an integrable map nearby?
- The problem is open.