Large values of class numbers of real quadratic fields

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- The number of imaginary quadratic fields with a given class number h is **finite**.
- There are infinitely many real quadratic fields with class number 1.

Imaginary quadratic fields

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Baker (1966), Heegner (1952) and Stark (1967)

There are exactly nine imaginary quadratic fields with class number 1, namely those corresponding to discriminants:

$$-3, -4, -7, -8, -11, -19, -43, -67, -163.$$

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For any given $\epsilon > 0$, there exists an effectively computable constant $c_\epsilon > 0$ such that

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Watkins (2004)

Determined the list of all imaginary quadratic fields with class number $h \leq 100$.

Why is it easier for imaginary vs real quadratic fields?

Dirichlet's class number formula (1839)

If $-d$ is a fundamental discriminant, then

$$h(-d) = \frac{\omega}{2\pi}\sqrt{d} \cdot L(1, \chi_{-d}),$$

where $\omega = 6$ if $d = 3$, $\omega = 4$ if $d = 4$ and $\omega = 2$ if $d > 4$.

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- The analogous class number formula for real quadratic fields is more complicated, due to the appearance of **non-trivial units** in this case.

How small (or large) can $L(1, \chi_d)$ be?

Classical Bounds

- $L(1, \chi_d) \ll \log |d|$.
- If $L(\sigma + it, \chi_d)$ has no zeros in the region $1 - c/\log(|d|(t+2)) < \sigma < 1$ then

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Siegel's Theorem (1931)

- For every $\epsilon > 0$, there exists a constant C_ϵ such that

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- The proof is not effective.

Conditional bounds for $L(1, \chi_d)$

Theorem 1 (Littlewood, 1928)

Assume the Generalized Riemann Hypothesis GRH. Then

$$(\zeta(2) + o(1))(2e^{\gamma} \log\log |d|)^{-1} \leq L(1, \chi_d) \leq (2e^{\gamma} + o(1)) \log\log |d|,$$

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Assume GRH.

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Assume GRH. If $-d < -4$ is a fundamental discriminant, then

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Let $3 \leq \tau \leq \log\log x + O(1)$.

- The number of imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ with $d \leq x$ such that $h(-d) \geq \frac{e^{\gamma}}{\pi}\sqrt{d} \cdot \tau$ is

$$|\mathcal{D}_{\text{im}}(x)| \cdot \exp\left(-\frac{e^{\tau - A}}{\tau}(1 + o(1))\right),$$

where $A := \int_0^1 \frac{\tanh(t)}{t} dt + \int_1^{\infty} \frac{\tanh(t)-1}{t} dt = 0.8187 \cdots$.

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Conjecture (Gauss, 1801)

There exist infinitely many positive discriminants d for which $h(d) = 1$.

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- By Dirichlet's Class Number Formula

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(a) There are **at least** $x^{1/2-1/\log\log x}$ real quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminant $d \leq x$, such that

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- Using **Heath-Brown's quadratic large sieve** and **zero-density estimates** for Dirichlet L-functions, we prove that Littlewood's GRH bound $L(1, \chi_d) \leq (2e^{\gamma} + o(1)) \log\log d$ holds (unconditionally) **for all but at most** $x^{1/2+o(1)}$ fundamental discriminants $0 < d < x$.

Chowla's family of real quadratic fields

- To prove part a) we use the following family of fundamental discriminants, first studied by Chowla:

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- For any h there are only **finitely** many real quadratic fields of Chowla's type with class number h.

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Theorem (Biró, 2003)

Chowla's conjecture is true.

The distribution of class numbers in Chowla's family

Theorem (Littlewood, 1928)

Assume GRH. If $d \in \mathcal{D}_{\text{ch}}$ then

$$\left(e^{-\gamma}\zeta(2)+o(1)\right)\frac{\sqrt{d}}{\log d \log\log d} \leq h(d) \leq \left(4e^{\gamma}+o(1)\right)\frac{\sqrt{d}}{\log d}\log\log d.$$

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- For $\tau \geq 1$, what is the proportion of discriminants $d \in \mathcal{D}_{\text{ch}}$ for which

$$h(d) \geq 2e^{\gamma}\frac{\sqrt{d}}{\log d}\cdot\tau, \text{ or } h(d) \leq \left(2e^{-\gamma}\zeta(2) + o(1)\right) \frac{\sqrt{d}}{\log d}\cdot\tau^{-1}?$$

Theorem (Dahl and L, 2016)

Let $1 \leq \tau \leq (1+o(1)) \log\log x$.

- The number of discriminants $d \in \mathcal{D}_{\text{ch}}(x)$ such that

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- The constant A is the **same** as in the result of Granville and Soundararajan for the distribution of class numbers of imaginary quadratic fields.

A probabilistic random model

Strategy

- Let $\mathcal{D}$ be a family of fundamental discriminants.
- "Construct a random Euler product"

$$L(1, \mathbb{X}) := \prod_{p} \left(1 - \frac{\mathbb{X}(p)}{p}\right)^{-1},$$

where $\mathbb{X}(p)$ are independent random variables taking the values $1, -1$ and 0 with the probabilities α_p, β_p and γ_p respectively.

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- Compare the distribution of $L(1, \chi_d)$ as d varies in $\mathcal{D}$ with that of the probabilistic model $L(1, \mathbb{X})$.

Granville and Soundararajan

If $\mathcal{D} = \mathcal{D}_{\text{im}}$, or $\mathcal{D}$ is the set of all fundamental discriminants, then $\alpha_p = \beta_p = p/(2(p+1))$ and $\gamma_p = 1/(p+1)$.

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where $c(p) := 1 + \left(\frac{-1}{p}\right)$ is the number of solutions of the congruence $4m^2 + 1 \equiv 0 \pmod{p^r}$, for any $r \geq 1$.

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- The slight "bias" in the distribution of $\mathbb{X}(p)$ towards the value -1 comes from the Jacobsthal sum identity

$$\sum_{m=1}^{p} \left(\frac{4m^2+1}{p}\right) = -1.$$

The number of fields with a given class number: Case I imaginary quadratic

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Conjecture (Soundararajan, 2007)

$$\frac{h}{\log h} \ll \mathcal{F}_{\text{im}}(h) \ll h \log h.$$

Conjecture (Holmin, Jones, Kurlberg, McLeman, Petersen, 2015)

As $h \to \infty$ we have

$$\mathcal{F}_{\mathrm{im}}(h) \sim \mathcal{C} \cdot c(h) \cdot \frac{h}{\log h}$$

where

$$\mathcal{C} = 15 \prod_{p>2} \prod_{i=2}^{\infty} \left(1 - \frac{1}{p^i}\right) \approx 11.317 \text{ and } c(h) = \prod_{p^n \| h} \prod_{i=1}^{n} \left(1 - \frac{1}{p^i}\right)^{-1}.$$

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Theorem (Soundararajan, 2007)

For large h we have

$$\mathcal{F}_{\mathrm{im}}(h) \ll h^2 \frac{(\log\log h)^4}{(\log h)^4}.$$

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- Guess: $\sum_{h \leq H} \mathcal{F}_{\mathrm{ch}}(h) \asymp H \log H$.

Theorem (Dahl and L, 2016)

$$\sum_{h \le H} \mathcal{F}_{\mathrm{ch}}(h) = \frac{1}{2G} H \log H + O\left(H (\log\log H)^3\right),$$

where

$$G = L(2, \chi_{-4}) = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \cdots = 0.916\ldots$$

is Catalan's constant, and χ_{-4} is the non-principal character modulo 4.

Thank you for your attention !