A counterexample on the reconstruction of oligomorphic clones

Manuel Bodirsky, David Evans, **Michael Kompatscher**, Michael Pinsker

michael@logic.at

Theory and Logic group
Technische Universität Wien

Homogeneous structures
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Reconstruction

Definition

We say a closed group $\Sigma \leq \mathrm{Sym}(\omega)$ has reconstruction iff for every $\Sigma' \leq \mathrm{Sym}(\omega)$:

$$\Sigma \cong \Sigma' \Rightarrow \Sigma \cong_T \Sigma',$$

where $\cong_T$ denotes topological isomorphicity in the topology of pointwise convergence.

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We similarly define reconstruction for

- transformation monoids in ω^ω
- function clones in $\bigcup_{n\geq 1} \omega^{\omega^n}$.

Clones

$\mathscr{C} \subseteq \bigcup_{n\geq 1} A^{A^n}$ is called function clone if

- it contains all projections $\pi_i^n(x_1, \ldots, x_n) = x_i$
- it is closed under composition $f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$

A map $\xi : \mathscr{C} \to \mathscr{D}$ is a clone homomorphism if

- ξ preserves arities
- $\xi(\pi_i^n) = \pi_i^n$
- it commutes with composition: $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$

$\bigcup_{n\geq 1} \omega^{\omega^n}$... topology of pointwise convergence.

ω-categorical structures

Ahlbrandt & Ziegler '86

Two countable ω-categorical structures $\mathcal{A}, \mathcal{B}$ are bi-interpretable iff

$$\mathrm{Aut}(\mathcal{A}) \cong_T \mathrm{Aut}(\mathcal{B}).$$

- What information is contained in $\mathrm{Aut}(\mathcal{A})$ as abstract group?
- Does every oligomorphic group have reconstruction?

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Bodirsky & Pinsker '12

Two countable ω-categorical structures $\mathcal{A}, \mathcal{B}$ are primitive positive bi-interpretable iff

$$\mathrm{Pol}(\mathcal{A}) \cong_T \mathrm{Pol}(\mathcal{B}).$$

- Does every oligomorphic clone have reconstruction?

Results

Evans & Hewitt '90

There is an ω-categorical structure $\mathcal{A}$, such that $\mathrm{Aut}(\mathcal{A})$ has no reconstruction.

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There is an ω-categorical structure $\mathcal{B}$ in *finite signature*, such that $\mathrm{Aut}(\mathcal{B})$, $\mathrm{End}(\mathcal{B})$ and $\mathrm{Pol}(\mathcal{B})$ have no reconstruction.

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These are results in ZFC.
ZF(+DC) is consistent with: "Every isomorphism between closed subgroups of $\mathrm{Sym}(\omega)$ is a homeomorphism."

Proof outline

1. There are profinite $G_1, G_2 \leq \mathrm{Sym}(\omega)$ such that $G_1 \cong G_2$ but $G_1 \not\cong_T G_2$.
2. Lift to oligomorphic $\Sigma_1, \Sigma_2 \leq \mathrm{Sym}(\omega)$ such that $\Sigma_1 \cong \Sigma_2$ but $\Sigma_1 \not\cong_T \Sigma_2$.
3. The topological closures $\overline{\Sigma_1}$ and $\overline{\Sigma_2}$ in ω^ω are isomorphic, but not topologically isomorphic.
4. The clone closures $\mathrm{Clo}(\overline{\Sigma_1})$, $\mathrm{Clo}(\overline{\Sigma_2})$ are isomorphic, but not topologically isomorphic
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The profinite group G

Fact (Witt '54)

There exists a separable profinite group G such that

- There is a finite central subgroup F such that $G = F \times E$ for an $E \leq G$.
- Every complement of a central subgroup is dense in G.

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Every isomorphism $F \times G/F \to G$ maps the image of the compact group $1 \times G/F$ to a dense subgroup of G. Therefore $G \not\cong_T F \times G/F$.

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Encoding profinite groups with oligomorphic groups

Proposition (Cherlin, Hrushovski)

For every separable profinite group R there is an oligomorphic Σ_R:

- There is a continuous homomorphism $\nu : \Sigma_R \to R$
- The kernel Φ is the intersection of open subgroups of finite index in Σ_R

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Wishful thinking

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$$G \not\cong_T F \times G/F \Rightarrow \Sigma_G \not\cong_T \Sigma_{F \times G/F}$$
$$G \cong F \times G/F \Rightarrow \Sigma_G \cong \Sigma_{F \times G/F}$$

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Problem: We do not know if $\Sigma_G \cong \Sigma_{F \times G/F}$ holds.

G/F and G as permutation groups

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How does this translate to permutation groups?

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- $G/F \to \mathrm{Sym}(\bigcup_{i\geq 1} G/G_i)$ is faithful and continuous.
- $F \times G/F \cong G$ gives us an action of G/F on G/G_0.

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- $F \times G/F \cong G$ gives us an action of G/F on G/G_0.
- $G/F \to \mathrm{Sym}(G/G_0 \cup \bigcup_{i\geq 1} G/G_i)$ is not continuous.
  The closure of the image of G/F is isomorphic to G.

From $\Sigma_{G/F}$ to Γ

Similarly start with $\Sigma_{G/F}$, oligomorphic on A.

- Let $\Sigma_{G/F}$ act on G/G_0 via $\nu : \Sigma_{G/F} \to G/F$.
- The action $i : \Sigma_{G/F} \to \mathrm{Sym}(A \cup G/G_0)$ is not continuous.
- Let Γ be the closure of $i(\Sigma_{G/F})$ in $\mathrm{Sym}(A \cup G/G_0)$.

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Result for groups

- Γ is oligomorphic on $A \cup G/G_0$,
- $\Gamma \cong i(\Sigma_{G/F}) \times F$,
- G is the quotient of Γ and the intersection of open subgroups with finite index.

Conclusion: $\Gamma \cong \Sigma_{G/F} \times F$ but $\Gamma \ncong_T \Sigma_{G/F} \times F$

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Problem: How to lift a *non-continuous* isomorphism $\Gamma \cong \Sigma_{G/F} \times F$ to the topological closure?!

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Lemma

The encoding homomorphism $\nu : \Sigma_{G/F} \to G/F$ extends to a continuous monoid homomorphism

$$\nu : \overline{\Sigma_{G/F}} \to G/F \text{ with kernel } \overline{\Phi}.$$

Lifting to the monoid closure

Define via $\nu : \overline{\Sigma_{G/F}} \to G/F$ an action of $\overline{\Sigma_{G/F}}$ on G/G_0. The combined action

$$i : \overline{\Sigma_{G/F}} \to (A \cup G/G_0)^{(A \cup G/G_0)}$$

is not continuous. The topological closure of $i(\overline{\Sigma_{G/F}})$ is $\overline{\Gamma}$.

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Result for monoids

$\overline{\Gamma}$ and $\overline{\Sigma_{G/F}} \times F$ are isomorphic, but not topologically isomorphic.

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The clone closure

Let $\mathrm{Clo}(\Lambda)$ denote the smallest clone containing the transformation monoid Λ. So

$$f(x_1, \ldots, x_n) \in \mathrm{Clo}(\Lambda) \Leftrightarrow f(x_1, \ldots, x_n) = g(x_i)$$

for $g \in \Lambda$, $1 \leq i \leq n$.

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Observation

Let $I:\Lambda\to\Delta$ be a monoid isomorphism. If I and I^{-1} send constants to constants, it has a natural extension to a clone isomorphism $\mathrm{Clo}(\Lambda)\to\mathrm{Clo}(\Delta)$.

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Result for clones

The clones $\mathrm{Clo}(\overline{\Gamma})$ and $\mathrm{Clo}(\overline{\Sigma_{G/F}} \times F)$ are isomorphic but not topologically isomorphic.

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Finite language

Proposition (Hrushovski)

Let $\mathcal{A}$ be ω-categorical. Then there is an ω-categorical $\mathcal{B}$ in a finite language including a predicate P, such that:

- the domain of $\mathcal{A}$ is equal to $P^{\mathcal{B}}$
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Idea

Let $\mathcal{A}$ be such that $\Sigma_{G/F} = \mathrm{Aut}(\mathcal{A})$. Take $\mathcal{B}$ given by Hrushovski's construction.

Make sure that $\mathrm{Aut}(\mathcal{B})$ still "encodes" G/F and $\mathrm{End}(\mathcal{B}) = \overline{\mathrm{Aut}(\mathcal{B})}$. Then $F \times \mathrm{Aut}(\mathcal{B})$ and $F \times \mathrm{End}(\mathcal{B})$ have no reconstruction.

Thank you!