Ultrafilters and structural Ramsey theory

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Homogeneous Structures
BIRS

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A typical class we consider is $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, where **D** is a countably infinite relational structure and $\mathrm{Age}(\mathbf{D})$ is the class of all finite structures embeddable into **D**.

Unless specified otherwise, **D** and $\mathcal{D}$ will be as above.

Let $\mathbf{A}$ and $\mathbf{B}$ be L-structures. An *embedding* $f : \mathbf{A} \to \mathbf{B}$ is an injective map preserving all relations and non-relations. We write $\mathrm{Emb}(\mathbf{A}, \mathbf{B})$ for the set of all such embeddings, and put $\mathbf{A} \leq \mathbf{B}$ if $\mathrm{Emb}(\mathbf{A}, \mathbf{B}) \neq \emptyset$.

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Let $\mathcal{D}$ be a class of finite L-structures. We say that $\mathbf{A} \in \mathcal{D}$ is a *Ramsey object* if for every $\mathbf{B} \in \mathcal{D}$ with $\mathbf{A} \leq \mathbf{B}$ and every $k \in \mathbb{N}$, there is a $\mathbf{C} \in \mathcal{D}$ with $\mathbf{B} \leq \mathbf{C}$ for which we have

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This says that for every coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to [k]$, there is $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ so that $|\gamma(f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}))| = 1$.

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We consider partial colorings $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{D}) \to [k]$; we say γ is *full* if $\mathrm{dom}(\gamma) = \mathrm{Emb}(\mathbf{A}, \mathbf{D})$, and we say γ is *large* if $\mathrm{dom}(\gamma)$ is thick.

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We also consider cofinal subclasses $\mathcal{C} \subseteq \mathcal{D}$, where *cofinal* refers to the embeddability partial order $\leq$.

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4. *For any large k-coloring* γ *of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$, *there is some* γ_i *which is thick.*

Proof.

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Assume $\neg$(2). Use compactness to construct a failure of (3). $\square$

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Let $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, where $\mathbf{D}$ is a countably infinite relational structure. Let $\mathbf{A} \in \mathcal{D}$, and write $H_{\mathbf{A}} := \mathrm{Emb}(\mathbf{A}, \mathbf{D})$. An ultrafilter $p \in \beta H_{\mathbf{A}}$ is called *thick* if every $T \in p$ is thick.

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1. *$\mathbf{A}$ is a Ramsey object,*
2. *There is a thick ultrafilter $p \in \beta H_{\mathbf{A}}$.*

Let $\mathbf{A} \leq \mathbf{B} \in \mathcal{D}$, and let $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$. We define the *dual map* $\hat{f} : H_B \to H_A$ via $\hat{f}(x) = x \circ f$.

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We can continuously extend $\hat{f}$ to $\tilde{f} : \beta H_{\mathbf{B}} \to \beta H_{\mathbf{A}}$. If $q \in \beta H_{\mathbf{B}}$ and $S \subseteq H_{\mathbf{A}}$, we have $S \in \tilde{f}(q)$ iff $\hat{f}^{-1}(S) \in q$.

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If $T \subseteq H_{\mathbf{A}}$ is thick, then $\hat{f}^{-1}(T)$ is thick. Fix $\mathbf{C} \in \mathcal{D}$ with $\mathbf{B} \leq \mathbf{C}$. Find $h \in H_{\mathbf{C}}$ with $h \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq T$. Then $h \circ \mathrm{Emb}(\mathbf{B}, \mathbf{C}) \subseteq \hat{f}^{-1}(T)$. So "thick moves up."

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If $\mathbf{A}$ and $\mathbf{B}$ are both Ramsey objects and $p \in \beta H_{\mathbf{A}}$ is thick, then there is thick $q \in \beta H_{\mathbf{B}}$ with $\tilde{f}(q) = p$.

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If **K** is a countably infinite relational structure with $\mathcal{K} = \mathrm{Age}(\mathbf{K})$, we call **K** a *Fraïssé structure* if **K** satisfies the *extension property*: given $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$, $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$, and $h \in \mathrm{Emb}(\mathbf{A}, \mathbf{K})$, then there is $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{K})$ so that $x \circ f = h$.

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Equivalently, using the shorthands $H_{\mathbf{A}}$ and $H_{\mathbf{B}}$ similar to before, $\mathbf{K}$ satisfies the extension property iff each dual map $\hat{f} : H_{\mathbf{B}} \to H_{\mathbf{A}}$ is surjective.

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The group $G = \mathrm{Aut}(\mathbf{K})$ acts on each $H_{\mathbf{A}}$ on the left; in Fraïssé structures this action is transitive.

Proposition

Let $\mathbf{K}$ *be a Fraïssé structure, with* $\mathcal{K} = \mathrm{Age}(\mathbf{K})$*. If* $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$*,* $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$*, and* $T \subseteq H_{\mathbf{B}}$ *is thick, then* $\hat{f}[T]$ *is thick.*

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Corollary

With notation as above, if $\mathbf{B}$ *is a Ramsey object, then* $\mathbf{A}$ *is a Ramsey object.*

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For $m \leq n$, denote by $i_m^n : \mathbf{A}_m \to \mathbf{A}_n$ and $i_n : \mathbf{A}_n \to \mathbf{K}$ the inclusion maps. Write $H_n := \mathrm{Emb}(\mathbf{A}_n, \mathbf{K})$.

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Here $n \geq m$ is large enough so that $g(\mathbf{A}_m) \subseteq \mathbf{A}_n$.

This right G-action is jointly continuous, where G is given the pointwise convergence topology. Let $1 \in \varprojlim \beta H_n$ be the element with $1(n) = i_n$. Then the orbit of 1 in $\varprojlim \beta H_n$ is dense, turning $(\varprojlim \beta H_n, 1)$ into a *G-ambit*.

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Fact: $(\varprojlim \beta H_n, 1)$ is the *greatest ambit*, meaning that for any other G-ambit (X, x_0), there is a continuous map $\varphi : \varprojlim \beta H_n \to X$ which respects the G-action and the distinguished points.

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A topological group G is said to be *extremely amenable* if the greatest G-ambit admits a fixed point.

Let $R_n \subseteq \beta H_n$ denote the closed subspace of thick ultrafilters. We say that $\mathcal{K}$ has the *Ramsey Property* if each object is a Ramsey object.

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Theorem (Kechris, Pestov, Todorcevic)

With $\mathbf{K}$, $\mathcal{K}$, *and* G *as before, then* G *is extremely amenable iff* $\mathcal{K}$ *has the Ramsey Property.*

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Pick large $n \geq m$, so $T_1 := \{x \in H_n : x \circ i_m^n \in S\}$ and $T_2 := \{x \in H_n : x \circ g|_m \notin S\}$ are in $\alpha(n)$.

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Thanks!