An Application of Model Theoretic Ramsey Theory

Caroline Terry

University of Illinois at Chicago

Homogeneous Structures Workshop
joint with M. Malliaris

Background

Definition

Suppose $G = (V, E)$ is a graph and $X \subseteq V$. The set X is called *complete* if every pair of elements in X are adjacent. The set X is called *independent* if every pair of elements in X are non-adjacent.

Theorem (Ramsey's Theorem)

For all $s \in \mathbb{N}$, there is a number $R(s) \in \mathbb{N}$ such that any graph with $R(s)$ vertices contains a complete or independent set of size s.

Fact (Bounds for R(s))

$$(1 + o(1))\frac{\sqrt{2}s}{e}2^{s/2} \leq R(s) \leq s^{\frac{-c\log s}{\log\log s}}4^s.$$

Ramsey-like graphs

In other words, there are constants C_1, C_2 such that for all n,

1. A graph G with n vertices always contains a complete or independent set of size $C_1 \log n$.
2. There are graphs of size n containing no complete or independent set bigger than $C_2 \log n$. Such graphs are constructed via probabilistic methods.

Definition

If a graph G has size n and no complete or independent set of size $C_2 \log n$, call G a *Ramsey-like graph*.

Erdős-Hajnal: "There are many ways to express that such graphs are complicated and similar to random graphs."

Questions

Questions

1. What does it mean to be “like random graphs”?
2. What does it mean to be “complicated”?

These are questions investigated by model theory in the infinite setting via notions of combinatorial complexity called *dividing lines*.

Model theoretic approach: Consider notions from the infinite setting which correspond to low complexity. Is there a way of finitizing them? Does doing so shed light on whats going on in the finite setting?

Example

Definition

Given an integer $k \geq 2$, a *half-graph of height k* is a graph G with vertex set $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ such that a_i is adjacent to b_j in G if and only if $i \leq j$.

Definition

A graph G is called *k-edge stable* if it contains no half-graph of height k as an induced subgraph.

Theorem (Malliaris-Shelah)

For every integer $k \geq 2$, there is an $\epsilon = \epsilon(k)$ such that for every n, a k-edge stable graph with n vertices has a complete or independent set of size n^ϵ.

Stable graphs are really un-Ramsey-like ($n^\epsilon >> C_2 \log n$).

Arranging graphs into type trees: some notation

Given $n \geq 2$, set

$$2^{<n} = \{ \text{ sequences of 0's and 1's of length } < n\}$$

Give $2^{<n}$ a tree structure as follows. Suppose $\eta, \eta' \in 2^{<n}$, set

$$\eta \triangleleft \eta' \Leftrightarrow \eta \text{ is a proper initial segment of } \eta'.$$

Given $\eta \in 2^{<n-1}$ and $i \in \{0, 1\}$, write $\eta \wedge i$ for the element of $2^{<n}$ obtained by adding i to the end of η.

Given $X \subseteq 2^{<n}$, think of X as a tree with structure induced from $2^{<n}$. A *branch* in X is a maximal comparable subset of X.

Arranging graphs into type trees

Suppose $G = (V, E)$ is a finite graph of size n. Here is an algorithm to index V by a subset $X \subseteq 2^{<n}$.

Level 0**:** Choose any $v_{<>} \in V$.

Level 1**:**
• If there are vertices adjacent to $v_{<>}$, choose one to be v_1.
• If there are vertices non-adjacent to $v_{<>}$ (other than $v_{<>}$), choose one to be v_0.

Level $k+1$**:** Suppose we have chosen our indexing up to level k. Choose some v_η in level k. Set $B_\eta = \{v_{\eta'} : \eta' \triangleleft \eta\}$.

Let $X_\eta = \{\text{vertices } v \text{ not yet chosen such that } N(v) \cap B_\eta = N(v_\eta) \cap B_\eta\}$.

• Choose $v_{\eta\wedge 1} \in N(v_\eta) \cap X_\eta$ (if possible).
• Choose $v_{\eta\wedge 0} \in \overline{N(v_\eta)} \cap X_\eta$ (if possible).

Continue this process until we've done this for all v_η from level k.

Arranging graphs into type trees

At the end of this algorithm, we end up with an indexing

$$V = \{v_\eta : \eta \in X\}$$

for some $X \subseteq 2^{<n}$ with the following properties:

1. For every $\eta \in X$, if $\eta \wedge 1 \in X$, then v_η is adjacent to $v_{\eta \wedge 1}$.
2. For every $\eta \in X$, if $\eta \wedge 0 \in X$, then v_η is non-adjacent to $v_{\eta \wedge 0}$.
3. For every η, if $\eta \wedge 1$ and $\eta \wedge 0$ are both in X then for every $\eta' \lhd \eta$,

$$v_{\eta \wedge 1} \text{ is adjacent to } v_{\eta'} \Leftrightarrow v_{\eta \wedge 0} \text{ is adjacent to } v_{\eta'}.$$

Call such an indexing $V = \{v_\eta : \eta \in X\}$ a *type tree*.

Long branches

Observation

Suppose $G = (V, E)$ is a graph with n vertices, $X \subseteq 2^{<n}$ and $V = \{v_\eta : \eta \in X\}$ is a type tree. Suppose η_1 is at the end of a branch B in X. Set $V_B = \{v_\eta : \eta \in B\}$. Then

(i) $\{v_{\eta_1}\} \cup (N(v_{\eta_1}) \cap V_B)$ is complete.

(ii) $\{v_{\eta_1}\} \cup \overline{N(v_{\eta_1})} \cap V_B$ is independent.

Proof.

We show (i). Suppose v_{η_2} and v_{η_3} are distinct points in $N(v_{\eta_1}) \cap V_B$. We want to show v_{η_2} and v_{η_3} are adjacent. Without loss assume $\eta_3 \lhd \eta_2 \lhd \eta_1$. Because the indexing is a type tree,

$$v_{\eta_1} \text{ is adjacent to } v_{\eta_3} \Leftrightarrow v_{\eta_2} \text{ is adjacent to } v_{\eta_3}.$$

$v_{\eta_3} \in N(v_{\eta_1}) \Rightarrow v_{\eta_1}$ is adjacent to $v_{\eta_3} \Rightarrow v_{\eta_2}$ is adjacent to v_{η_3}. □

Long branches

Corollary

Suppose $G = (V, E)$ is a graph with n vertices, $X \subseteq 2^{<n}$ and $V = \{v_\eta : \eta \in X\}$ is a type tree. Suppose X contains a branch of length h. Then G contains a complete or independent set of size $h/2$.

Proof.

Suppose B is a branch in X of size h with end vertex v_{η_1}. Then

$$V_B = \Big(\{v_{\eta_1}\} \cup (N(v_{\eta_1}) \cap V_B)\Big) \sqcup \Big(\overline{N(v_{\eta_1})} \cap V_B\Big),$$

(i) $\{v_{\eta_1}\} \cup (N(v_{\eta_1}) \cap V_B)$ is complete, and

(ii) $\overline{N(v_{\eta_1})} \cap V_B$ is independent.

One must have size $|V_B|/2 = |B|/2 = h/2$. □

Tree rank and tree height

Question

What can force graphs to have long branches?

Definition

Suppose $G = (V, E)$ is a graph. The *tree rank* of G, $t = t(G)$, is the largest integer such that there is a set $V' \subseteq V$ and an indexing $V' = \{v_\eta : \eta \in 2^{<t}\}$ which is a type tree.

Idea: $t(G) =$ largest possible "full binary tree" inside G.

Definition

Suppose $G = (V, E)$ is a graph. The *tree height* of G, $h = h(G)$, is the smallest integer such that every indexing of V which is a type tree has a branch of length h.

Tree rank and tree height

Theorem (Malliaris-T., adapted from Malliaris-Shelah)

If a graph G has N vertices, tree rank t and tree height h, then

$$h \geq \frac{(N/t)^{\frac{1}{t+1}}}{2}.$$

Idea: tree can't get too full/wide, so forces a long branch to happen.

Definition

A family of graphs $\mathcal{G}$ is *tree-bounded* if there is an integer t such that for all $G \in \mathcal{G}$, $t(G) \leq t$.

Tree-bounded families

Corollary

If $\mathcal{G}$ is a collection of finite graphs which is tree bounded, then there is an ϵ such that every N-element $G \in \mathcal{G}$ contains a complete or independent set of size

$$N^{\epsilon} >> C_2 \log N.$$

Tree-bounded families are really un-Ramsey-like.

Examples

The following $\mathcal{G}$ are tree-bounded families.

1. Given $k \geq 2$, $\mathcal{G} =$ the family of finite k-edge stable graphs.
2. Given $n \geq 2$, $\mathcal{G} =$ the family of K_n-free graphs.
3. Given any finite graph H which can be realized as a branch through a type tree, $\mathcal{G} =$ the family of finite H free graphs.

Note: the second family has unbounded VC-dimension.

For an application of this improved Ramsey's theorem to a theorem in structural graph theory, see

`http://arxiv.org/abs/1511.02544`

Thank you for listening!