Equimorphy versus isomorphy

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Introduction and motivation

Two structures are equimorphic if each embeds in the other. If they are infinite, these structures do not need to be isomorphic. Two conjectures relating the notions of equimorphy and isomorphy for infinite structures are the motivation for this talk.
I will present some results obtained with Claude Laflamme, Norbert Sauer and Robert Woodrow during the last five years.
Three parts:
1)**Bonato-Tardif's conjecture on trees**.
2) **Thomassé's conjecture on relational structures.**
3) **The hypergraph of copies of a countable homogeneous structure.**

Bonato-Tardif 's conjecture on trees.

Here, trees are connected (undirected) graphs without any cycle.
The tree alternative property holds for a tree T if either every tree equimorphic to T is isomorphic to T or there are infinitely many pairwise non-isomorphic trees which are equimorphic to T.
Bonato and Tardif (2006) conjectured that the tree alternative property holds for every tree.
They proved that it holds for rayless trees(trees not containing an infinite path, also called a ray).

A. Bonato, C. Tardif, Mutually embeddable graphs and the tree alternative conjecture. J. Combin. Theory Ser. B 96 (2006), no. 6, 874–880.

Tyomkyn (2009) proved that the tree alternative property holds for rooted trees.
Halin (1990) showed that every rayless tree contains a vertex or an edge preserved by every embedding.
This two results yield an other proof that the tree alternative property holds for rayless trees.
Tyomkyn conjectured that every locally finite tree T distinct from he infinite path which has a non-surjective embedding has infinitely many trees equimorphic to it.

M. Tyomkyn, A proof of the rooted tree alternative conjecture, Discrete Math. 309 (2009) 5963-5967.

R. Halin, Fixed configurations in graphs with small number of disjoint rays, in R.Bodendiek (Ed) Contemporary Methods in Graph Theory, Bibliographisches Inst., Mannheim, 1990, pp. 639-649.

Scattered trees.

A tree is scattered if no subdivision of the complete binary tree is a subtree. Claude Laflamme, Norbert Sauer and I (2015) proved that the tree alternative property holds for scattered trees and that Tyomkin conjecture holds for locally finite scattered trees.

C.Laflamme, M.Pouzet, N.Sauer, Invariant subsets of scattered trees. An application to the tree alternative property of Bonato and Tardif, 43pp, ArXiv, August 5, 2015.

Central to the argument is a fixed set property of scattered trees. An end in a tree is an equivalence class of ray. A tree is scattered iff the space of ends is topologically scattered (Jung, 1969, Polat, 96). Hence every end has a (Cantor-Bendixson) rank.

Theorem

If a tree T is scattered then either there is one vertex, one edge, or a set of at most two ends preserved by every embedding of T.

Ou result extends a result of Polat and Sabidussi (1994). They proved that a scattered tree having a set of at least three ends of maximal rank contains a rayless tree preserved by every automorphism.

N. Polat, G. Sabidussi, Fixed elements of infinite trees. Graphs and combinatorics (Lyon, 1987; Montreal, PQ, 1988). Discrete Math. 130 (1994), no. 1-3, 97–102.

Our contribution in the theorem above is about scattered trees for which there is no end of maximal rank. We prove that if T is such a tree then it contains a rayless tree preserved by every embedding of T.

The existence of a set of two ends preserved by every embedding is quite easy to characterize:

Proposition

A tree T has a set of two distinct ends preserved by every embedding if and only if it contains a two-way infinite path preserved by every embedding.

The case of an end preserved by every embedding is more subtle. Indeed, this does not imply that the end contains a ray preserved by every embedding. There are several example of scattered trees with such an end but which do not contain an infinite path nor a vertex nor an edge preserved by every embedding. Let T be a tree and e be an end. We say that e is preserved *forward*, resp. *backward*, by an embedding f of T if there is some ray $C \in e$ such that $f[C] \subseteq C$, resp. $C \subseteq f[C]$. We say that e is *almost rigid*if it is preserved backward and forward by every embedding, that is every embedding fixes pointwise a cofinite subset of every ray belonging to e.

If $C := \{x_0, \dots x_n, \dots\}$ is a ray, then T decomposes as a sum of rooted trees indexed by C, that is $T = \bigoplus_{x_i \in C} T_{x_i}$ where T_{x_i} is the tree, rooted at x_i, whose vertex set is the connected component of x_i in $T \setminus \{x_{i-1}, x_{i+1}\}$ if $i \geq 1$ and in $T \setminus \{x_{i+1}\}$ if $i = 0$. If the number of pairwise non-equimorphic rooted trees T_{x_i} is finite, we say that C is *regular*. We say that an end is *regular* if it contains some regular ray (in which case all other rays that it contains are regular).

Theorem

Let T be a tree. If T is scattered and contains exactly one end e of maximal rank then e is preserved forward by every embedding. If T contains a regular end e preserved forward by every embedding, then e contains some ray preserved by every embedding provided that T is scattered or e is not almost rigid.

Theorem

(i) *If a scattered T does not contain a vertex or an edge preserved by every embedding or an almost rigid end and has a non-surjective embedding then $|twin(T)| = \infty$ unless T is the one-way infinite path.*

(ii) *If T has an almost rigid end then $|twin(T)| = 1$ if and only if $|twin(T(\to x))| = 1$ for every vertex x, otherwise $|twin(T)| = \infty$.*

(iii) *T has a non-regular and not almost rigid end preserved forward by every embedding then $|twin(T)| \geq 2^{\aleph_0}$.*

Note that this result and the previous one imply:

Corollary

Let T be a scattered tree with $|twin(T)| < 2^{\aleph_0}$. Then there exists a vertex or an edge or a two-way infinite path or a one-way infinite path or an almost rigid end preserved by every embedding of T.

Stephan Thomassé, Conjectures on countable relations, 17 pages, circa 2000.

Conjecture. Let R be a countable relation. Is the number of relations equimorphic to R (counted up to isomorphy) is 1, $\aleph_0$ or $2^{\aleph_0}$?.

Unsolved for graphs, even in the case of loopless undirected graphs; in fact it is unsolved for trees.

Thomassé's conjecture and Bonato-Tardif conjecture.

Bonato-Tardif conjecture was extended to graphs and proved true for rayless graphs by Bonato et al, (2011).

A. Bonato, H. Bruhn, R. Diestel, P. Sprüssel, Twins of rayless graphs. J. Combin. Theory Ser. B 101 (2011), no. 1, 60–65.

Theorem

If the extension of B-T-conjecture holds then Thomasse's conjecture holds for graphs.

In fact, let $sib(G)$ be the number of graphs which are equimorphic to G, these graphs being counted up to isomorphy; similarly, if G is connected, let $sib_{conn}(G)$ be the number of those which are connected.

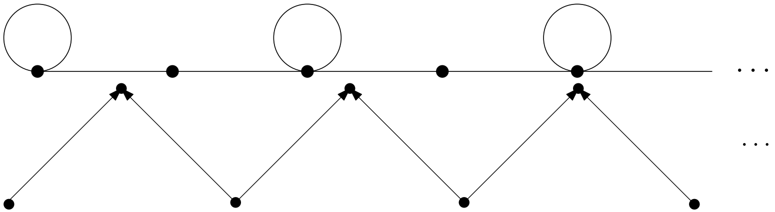
Theorem

Let G be a graph. Then $sib(G) = 1$ or is infinite, provided that $sib_{conn}(H) = 1$ or is infinite for every connected component H of G. Furthermore, if G is countable, then $sib(G) = 1$ iff

1. *for each connected component H, $sib_{conn}(H) = 1$*
2. *Each nontrivial connected component occurs only finitely many times in G;*
3. *There is no strictly increasing sequence (wrt embeddability) of connected components.*
4. *The set $Bad(G)$ of connected components H such that $sib(H) > 1$ is finite.*
5. *If $Bad(G) \neq \varnothing$ then G has countably many isolated vertices and for each $H \in Bad(G)$, every H' which is equimorphic to H is of the form $H \oplus \overline{K}_n$, where $\overline{K}_n$ is an independent set on n vertices with $n < \omega$.*

Problems and Comments

Find an example with $sib(G) = 1$ and Bad(G)$\neq \varnothing$. Note that for a connected graph H, $sib_{conn}(H) \neq sib(H)$ amounts to H equimorphic to H with an isolated vertex added, that is H equimorphic to $H \oplus 1$.
Find an example of connected graph H such that $1 = sib_{conn}(H) \neq sib(H)$ and every equimorphic H' is of the form $H \oplus \overline{K}_n$ with $n < \omega$.
Note that the extension of B-T conjecture to binary relations is false. In fact it is false for undirected graphs with loops and for ordered sets.



The case of Chains

In

C.Laflamme, M.Pouzet, R.Woodrow, Equimorphy-The case of chains, we proved that Thomassé's conjecture holds for chains.
Let $sib(C)$ be the number of chains which are equimorphic to C, those chains being counted up to isomorphy.

Theorem

If C is any chain, then $sib(C) = 1$ or is infinite; if C is countable then $sib(C) = 1, \aleph_0$ or $2^{\aleph_0}$.

Let us start with a simple fact:

Proposition

If C is a finite sums of ordinals and reverse ordinals, then $sib(C) = 1$.

Converse false: $sib(\omega.\omega^* + \omega^2) = 1$.

Behind the argument of the previous proposition lies the notion of well quasi ordering.

Proof.

Let n be the least integer such that C has a decomposition as a sum of n ordinals or reverse ordinals. Choose a decomposition $C := \sum_{i<n} C_i$ minimal in the sense that if $C := \sum_{i<n} C'_i$ is an other decomposition with $C'_i \leq C_i$ for $i < n$ then C'_i is equimorphic to C_i for all $i < n$; this exists since ordinals are well ordered under embeddability.

Now consider any chain $C' \equiv C$. Since $C' \leq C$, C' must be of the form $C' := \sum_{i<n} C'_i$ with $C'_i \leq C_i$ for all $i < n$. Since $C \leq C'$, the same argument yields that $C := \sum_{i<n} C''_i$ with $C''_i \leq C'_i$ for all $i < n$. Since $C'_i \leq C_i$ we have $C''_i \leq C_i$. From the minimality of the decomposition of C, we have $C''_i \equiv C_i$ hence $C'_i \equiv C_i$. This yields $C'_i \simeq C_i$ thus $C' \simeq C$. □

Scattered chains and surordinals

To go further, we need the notion of scattered chain (chain not embedding the chain of rational numbers) and use Hausdorff theorem: Every chain is a lexicographical sum of scattered chains indexed by a dense chain (possibly reduced to a singleton).
Scattered chains with few ($< 2^{\aleph_0}$) siblings are finite sums of surordinals and their reverse.
A chain C or its order type is a surordinal if for each $x \in C$ the cofinal segment generated by x is well ordered. Equivalently, $1 + \omega^*$ does not embed into C.

P. Jullien, Contribution à l'étude des types d'ordres dispersés, Thèse Doctorat d'État, Université de Marseille, 27 juin 1969, 116p.

Theorem

Let C be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1. $sib(C) = \kappa$ *and C is scattered;*
2. $\kappa = 1$, *or* $\kappa \geq \aleph_0$ *and C is a finite sum of surordinals and of reverse of surordinals, and if* $C = \sum_{j<m} D_j$ *is such a sum with m minimum then* $max\{sib(D_j) : j < m\} = \kappa$.

There are uncountable dense chains C such that $sib(C) = 1$, and one such construction is owed to Dushnik and Miller.

Lemma

If $f : C \to C$ is an order preserving map, and for some $x \in C$ the interval determined by x and $f(x)$ is non-scattered, then $sib(C) \geq 2^{\aleph_0}$.

Theorem

Let C be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1. $sib(C) = \kappa$.
2. $C = \sum_{i \in D} C_i$, *where:*
    - *D is dense (singleton or infinite),*
    - *each C_i is scattered,*
    - *$sib(C_i) = 1$ for all but finitely many $i \in D$,*
    - *$\max\{sib(C_i) : i \in D\} = \kappa$, and*
    - *every embedding $f : C \to C$ preserves each C_i.*

We recall that a permutation group G acting on a set E is oligomorphicif for every integer n there are only finitely many orbits of n-elements subsets of E. As it is well-known, if E is countable this amounts to the fact that $\overline{G}^{\mathfrak{S}}$ is the automorphism group of an $\aleph_0$-categorical theory.
C.Laflamme, N.Sauer, R.Woodrow and I proved:

Theorem

Let R be a countable homogeneous structure such that $Aut(R)$ is oligomorphic, then

1. *$sib(R)$ is either one or infinite.*
2. *$sib(R) = 1$ and only if R is finitely partitionable (that is there is a partition of the domain E of R into finitely many sets such that every permutation of E which preserves each block of the partition is an automorphism of R).*

This result is in fact about $Aut(R)$, the automorphism group of R. Indeed, let G be a group acting on a set E. We recall that a partial map f with domain A and codomain A' included into E is *adherent* to G w.r.t. the pointwise convergence topology if for every finite subset F of A there is some $g \in G$ such that f and g coincide on F. We say that such a map is a G-local embedding. If $A = E$ we say that this is a G-embedding and if furthermore $A' = E$ we say that this is a G-automorphism. The set of G-automorphisms forms a group, which is in fact $\overline{G}^{\mathfrak{S}}$, the adherence of G into $\mathfrak{S}(E)$, the groups of all permutations of E, equipped with the topology of pointwise convergence. We say that two subsets of E are *equivalent*, resp. *weakly-equivalent* if one is the image of the other by some G-local embedding, resp. each one contains the image of the other by somme G-local embedding. A G-copyis the image of E by some G-embedding, that is a member of the equivalence class of E. A G-sibling is a subset of E which contains a G-copy; equivalently, this is a subset weakly equivalent to E. We denote by $sib(G)$ the number of equivalence classes of G-siblings. We may rewrite our result as follows:

Theorem

If G is oligomorphic and E countable then $sib(G)$ is one or infinite. That is either the weakly-equivalence class of E coincide with the equivalence class of E (that is the set of copies) or it is the union of infinitely many equivalence classes. In the first case there is a partition of E into finitely many sets such that every permutation of E which preserves each block of the partition belongs to $\overline{G}^{\mathfrak{S}}$.

There are two aspects: the fact that $sib(G)$ is one or infinity, the characterization of G such that $sib(G) = 1$. We think that the first aspect is relevant of descriptive set theory and could have a simple proof (eg with some use of Galvin-Prikry theorem). The second uses Frasnay's theory, with a deep result, which is quite unescapable.

A key tool for the proof is the notion of monomorphic decomposition of a relational structure introduced in Pouzet-Thiéry, also studied in the thesis of D.Oudrar (2015) and a result of Frasnay.

M. Pouzet, N. Thiéry, Some relational structures with polynomial growth and their associated algebras I. Quasi-polynomiality of the profile, The Electronic J. of Combinatorics, 20(2) (2013), 35pp.

C.Frasnay, Chainable relations, rangements and pseudorangements. Orders: description and roles (L'Arbresle, 1982), 235–268, North-Holland Math. Stud., 99, North-Holland, Amsterdam, 1984. This result is in fact about $Aut(R)$, the automorphism group of R.

We attach to each relational structure R with domain E an equivalence relation on E, whose classes are the monomorphic components of R.

Case 1. The number of classes is finite. We guess that in this case $sib(R)$ is 1 or infinity. We obtain this conclusion, in fact $sib(R)$ is 1 or $\geq 2^{\aleph_0}$, when R is homogeneous. For that we use Frasnay's theory and the Cameron's result on monomorphic groups.

Case 2. The number of classes is infinite. We suppose that R is universal for its age. That is every countable R' with $Age(R') \subseteq Age(R)$ is embeddable into R. (this is the case if $Aut(R)$ is oligomorphic). Hence, every extension R' of R with the same age is equimorphic to R.

We built an extension R' of R to a superset E' of E such that $Age(R') = Age(R)$, $E' \setminus E$ is infinite and is contained into a class of the decomposition of R'. To every finite subset H of $E' \setminus E$, we associate the restriction $R_H := R'_{\restriction E \cup H}$. Then R_H is equimorphic to R. We prove that with a well choosen R', different sized sets H will give non-isomorphic R_H. Hence $sib(R)$ will be infinite.

With the assumption that $Aut(R)$ is oligomorphic, we will get a bound, say s, on the size of the finite monomorphic components. If R has no infinite monomorphic component, then for diffferent $|H|$ s the R_H's will be pairwise non isomorphic, this without any more assumption of $E' \setminus E$. For the general case we construct R' such that $E' \setminus E$ is an monomorphic class of R'. For that, we observe that every infinite monomorphic component is included into some orbit and we consider the two possible cases: 1) Some orbit of $Aut(R)$ is the union of infinitely many infinite monomorphic component; 2) Some infinite orbit is covered by infinitely many finitely sized monomorphic classes.
With compactness theorem and (ordinary) Ramsey theorem we construct R'.

Monomorphic decomposition of a relational structure

Let R a relational structure on a set E. A subset E' of E is a monomorphic part of R if for every integer k and every pair A, A' of k-element subsets of E, the induced structures on A and A' are isomorphic whenever $A \setminus E' = A' \setminus E'$. A monomorphic decomposition of R is a partition $\mathscr{P}$ of E into monomorphic parts. A monomorphic part which is maximal for inclusion is a monomorphic component of R.
The monomorphic components of R form a monomorphic decomposition of R of which every monomorphic decomposition of R is a refinement (Proposition 2.12 of Pouzet-Thiéry).
This partition was defined in a direct way by Oudrar and I.

Let x and y be two elements of E. Let F be a finite subset of $E \setminus \{x, y\}$, we say that x and y of are F-equivalentand we set $x \simeq_{F,R} y$ if the restrictions of R to $\{x\} \cup F$ and $\{y\} \cup F$ are isomorphic. Let k be a non-negative integer, we set $x \simeq_{k,R} y$ if $x \simeq_{F,R} y$ for every k-element subset F of $E \setminus \{x, y\}$. We set $x \simeq_{\leq k,R} y$ if $x \simeq_{k',R} y$ for every $k' \leq k$ and $x \simeq_R y$ if $x \simeq_{k,R} y$ for every k. This defines three equivalence relations on E. As it turns out:

Lemma

The equivalence classes of $\simeq_R$ are the components of R.

With Thiéry and I we proved (Theorem 2.25 of our 2013 paper):

Theorem

On each infinite equivalence class, say A, there is a linear order such that every local isomorphism of this linear order, once extended by the identity on $E \setminus A$, is a local isomorphism of R.

Relational structures with a finite monomorphic decomposition

From the theorem above it follows that a relational structure R has a finite monomorphic decomposition iff its domain E can totally ordered and divided into finitely many intervals $E_1, \ldots E_n$ such that every partial isomorphism f preserving the order on each interval preserves R.
Could we easily prove that in this case $sib(R)$ is 1 or infinity?
If there is just one class, R could be a chain. In this case, we obtained this conclusion. But if R is not a chain? Such R's are said chainable.
The relationship between two linear orders L_1, L_2 chaining the same infinite structure R was studied by Frasnay in 1965. He associated to such pair $B := (E, L_1, L_2)$ of linear orders an infinite sequence of finite permutation groups $G_m(B)$, $m \in \omega$, the indicative sequence. And he described the possible types of sequences.

A later result of Frasnay (a Lemma page 263, 1984) asserts that the indicative sequence of a pair $B := (E, L_1, L_2)$ st one of the linear orders has no extreme elements is one of five possible types.
According to a Cameron's result, this is the group sequence of a homogeneous monomorphic countable structure. (We recall that a permutation group G is monomorphic (or highly homogeneous) if two finite sets of he same size are in the same orbit.
We have:

Theorem

Le R be a countable structure such that $Aut(R)$ is monomorphic then R and $Aut(R)$ have the same number of siblings. Furthermore this number if 1 if $Aut(R)$ is the full permutation group, $2^{\aleph_0}$ otherwise.

If R has finitely many monomorphic components, getting the same conclusion requires some work; a first reduction is to prove that we may suppose that all components are infinite. In fact, the union of finite components forms the kernel $\ker(R)$ of R, where $\ker(R) := \{x \in R : Age(R_x) \neq Age(R)\}$. Hence every equimorphic copy of R contains R and has the same kernel.

An Extension (to be checked by L.S.W)

Delete the hypothesis that R is homogeneous. Keep the fact that $G := Aut(R)$ is oligomorphic. This amounts to the fact that R is $\aleph_0$ categorical. In this case, R is equimorphic to a uniformly prehomogeneous structure R' such that $Aut(R')$ is oligomorphic (Saracino 1973, I, 1972). Thus $sib(R) = sib(R')$. Hence, we only need to obtain the conclusion of our theorem for uniformly prehomogeneous structures.

A structure R' is uniformly prehomogeneous if for every finite set F' of the domain E' of R' there is a finite superset F'' of F' whose cardinality in bounded by some function θ of the cardinality of F' such that every local isomorphism of R' defined on F' extend to an automorphism provided that it extends to F'' (Pabion 1972).

As a consequence, will follows a result of Hodkinson and Macpherson (1988): a countable structure R in a finite language is such that every R' with the same age is isomorphic to R if and only if R is finitely partitionable (as they indicate, this conclusion holds if the language is infinite and in addition $Aut(R)$ oligomorphic).

Revising the tools

Let R be a relational structure with base E. A relational structure R' is a 1-*extension of* Rif its base E' contains E and for every finite subset H de E, every finite subset H' of $E' \setminus E$ there is a local isomorphism of R' to R which is the identity on H and maps H' into E.

Claim

-Si R' is a 1*-extension of* R *de base* E' *then the monomorphic decomposition of* R *is induced by the decomposition of* R'*.*
-If $Aut(R)$ *is oligomorphic then every extension* R' *with the same age is a* 1*-extension.*
-Let S *be the set of non-negative integers* k *such that* R *has some monomorphic component of size* k*. If* $\setminus S$ *is infinite and if* R *has a* 1*-extension* R' *such that* $E' \setminus E$ *is an infinite monomorphic component of* R' *then* R *has an infinitely many* 1*-extensions which are pairwise non isomorphic.*

Lemma

Let R be uniformly prehomogeneous on a countable set E, suppose that $Aut(R)$ is oligomorphic and R has infinitely many monomorphic components then there is an extension R' with the same age to an set E' such that $E' \setminus E$ is an component of R'.

Use the diagram's method due to Robinson and apply the compactness theorem of first order logic.

Case 1. One orbit, say A, contains infinitely many components Case 2. One orbit is covered by infinitely many components of the same size. Here use Ramsey's theorem to add an infinite monomorphic block.

The hypergraph of copies

Let R and R' be two relational structures whith domains E and E'. Denote by $Emb(R, R')$ the set of embeddings f of R into R' and by $Iso(R, R')$ the set of ranges of $f \in Emb(R, R')$, that we call the copies of R into R'. If $R = R'$, denote these sets by $Emb(R)$ and $Iso(R)$ respectively. We are interested in the properties of these sets particularly when R is homogeneous.

In this later case, everything can be expressed in terms of permutation groups. Indeed, to a permutation group G on a set E we associate $\overline{G}$, the topological closure of G into the power E^E equipped with the convergence topology and the set $I(\overline{G})$ of the ranges of $f \in \overline{G}$. Then if R is a homogeneous relational structure on E such that $Aut(R) = \overline{G}$ $:= \overline{G} \cap \mathfrak{S}(E)$ one has $Emb(R) = \overline{G}$ and $Iso(R) = I(\overline{G})$.

A sample of results

We suppose that E is countable. Our results fall into two parts.
Part 1 The group G is *oligomorphic*

Theorem

If G is oligomorphic there is an embedding from $\wp(\omega)$, the power set of ω ordered by inclusion, into $I(\overline{G})$.

Furthermore, define the *algebraic closure* $acl_G(X)$ of a subset X of E as the union of finite orbits of singletons of the stabilizer $G < X >$ of X if X is finite, otherwise set $acl_G(X) := \bigcup\{acl_G(F) : F \in [X]^{<\omega}\}$. Note that if G is oligomorphic then $acl_G(X)$ is finite for every finite subset X of E.

Theorem

If G is oligomorphic then for every finite subset X of E, $acl_G(X)$ is the intersection of two copies.

Problem

Does $acl_G(X)$ is an intersection of copies if X is infinite?

The proof of the theorems above relies on the use of Ehrenfeucht-Mostowski models. If G is not oligomorphic, there are no longer available.

Part 2 The group is not necessarily oligomorphic. A characterization of members of $I(\overline{G})$ is easy.

Proposition

A subset A of E belongs to $I(\overline{G})$ iff for every $F \in [A]^{<\omega}$ and every $x \in E \setminus F$ there is some $g \in G$ such that $g_{\restriction F} = id_{\restriction F}$ and $g(x) \in A$.

As a consequence we get:

Corollary

$I(\overline{G})$ is closed under the union of non-empty up-directed families; topologically, $I(\overline{G})$ is a G_δ-set of $\wp(E)$ equipped with the product topology. Moreover, if $I(\overline{G})$ has more than one element, it has no isolated point.

From this, we have:

Theorem

The cardinality of $I(\overline{G})$ is either 1 or $2^{\aleph_0}$.

Problem

Is it true that if $I(\overline{G})$ has more than one element then it is not totally ordered by inclusion? Is it true that it embeds $\wp(\omega)$?

We introduce a notion of rank for orbits of singletons over finite subsets of E. We define the *ranked closure* $rcl_G(X)$ of a subset X of E as the union of orbits of singletons of the stabilizer $G < X >$ of X which are ranked if X is finite, and otherwise we set $rcl_G(X) := \bigcup\{rcl_G(F) : F \in [X]^{<\omega}\}$. Notice that if G is oligomorphic, then the ranked closure coincide with the algebraic closure.

Theorem

The ranked closure of a finite set is an intersection of copies.

Corollary

There is just one copy iff the ranked closure of the empty set is E.

However, there are examples for which the ranked closure of the empty set is empty and still there are no pairwise disjoint copies. Let $T_{\mathbb{Z}}$ be the ordered tree on a countable set E in which each interval is finite, every element has countably many successors and a predecessor. Let $G := Aut(T_{\mathbb{Z}})$.

Proposition

$\wp(\omega)$ is embeddable in $I(\overline{G})$. Any two copies pairwise intersect; for every $x \in E$, $\downarrow x$ is the intersection of copies containing x.

We illustrate the notion of rank with automorphisms of chains.

Theorem

Let $G := Aut(C)$ where C is a countable chain. Then $rcl_G(\varnothing)$ is the union of scattered orbits of G. Furthermore, if $rcl_G(\varnothing) \neq E$ then $\wp(\omega)$ is embeddable into $I(\overline{G})$.

A group is *locally finite* is the algebraic closure of every finite subset of E is finite. Oligomorphic groups are locally finite. There are many others.

Theorem

If G is locally finite then for every finite subset X of E, $acl_G(X)$ is the intersection of two copies.

Problem

If G is locally finite, is it true that if $I(\overline{G})$ embeds $\wp(\omega)$?

Let $H_{\overline{G}} := (E, I(\overline{G}))$; we call this pair the *hypergraph* of copies.
We have the inclusion $\overline{G}^{\mathfrak{S}} \subseteq Aut(H_{\overline{G}})$ and a natural embedding from $Aut(H_{\overline{G}})$ into $Aut(I(\overline{G}))$.
Looking at examples, we show that $\overline{G}^{\mathfrak{S}} = Aut(H_{\overline{G}})$ if $\overline{G}^{\mathfrak{S}} = Aut(R)$ where R is either the countable homogeneous triangle free graph Γ or $\mathbb{B}$, the set of rational numbers equipped with the betweeness relation. On the other hand we show that $\overline{G}^{\mathfrak{S}} \neq Aut(H_{\overline{G}})$ if R is the Rado graph.

The natural embedding of $Aut(H_{\overline{G}})$ into $Aut(I(\overline{G}))$ is not always surjective. It is surjective if the algebraic closure of every finite set is finite.

Problem

Find a complete characterization.

THANK YOU FOR YOUR ATTENTION