Hrushovski property for homogeneous directed graphs

Homogeneous Spaces - BIRS 2015

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The Main Question

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Question 0 ($\mathcal{F}$). (Hrushovski, '92)

Let $\mathcal{F}$ be a Fraïssé class. Suppose the enemy gives you $\mathbb{A} \in \mathcal{F}$ a (medium) structure, and I a collection of partial automorphisms of $\mathbb{A}$. Is there a (large) structure $\mathbb{A} \leq \mathbb{B} \in \mathcal{F}$ such that each automorphism in I extends to an automorphism of $\mathbb{B}$?

Notice that $\mathrm{Flim}(\mathcal{F})$ would witness this if we didn't insist that $\mathbb{B}$ is finite.

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Question 1 ($\mathcal{F}$). (Hrushovski, '92)

What if I is just a single partial automorphism?

History

Theorem

The following classes have the Hrushovski property.

1. Pure sets - [Folklore].
2. Graphs - [Hrushovski 1992].
3. r-uniform hypergraphs - [Herwig 1995], K_n-free graphs - [Herwig 1998].
4. Various general "model theoretic" classes [Herwig-Lascar 2000, and others].
5. Metric spaces (with distances in a countable additive subsemigroup of $\mathbb{R}^+$) - [Solecki 2005]. Also [Vershik 2008], [Rosendal 2011], [Pestov] and [Sabok].
6. $\mathcal{T}[\mathbb{I}_n]$ and $\mathbb{I}_n[\mathcal{T}]$ if and only if $\mathcal{T}$, the class of tournaments, is - [P-Sokic 2015].

Connections with $\mathrm{Aut}(\mathrm{Flim}(\mathcal{F}))$ - Part 1.

Theorem (Kechris-Rosendal 2007)

Let $\mathcal{K}$ be a Fraïssé class (of finite structures). Then $\mathcal{K}$ has the Hrushovski property if and only if there is a countable chain $C_0 \leq C_1 \leq \ldots \leq \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ of compact subgroups whose union is dense in $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$.

This property is referred to as being *compactly approximable*.

Connections with $\mathrm{Aut}(\mathrm{Flim}(\mathcal{F}))$ - Part 2.

Suppose that $\mathcal{F}$ has a precompact expansion $\mathcal{F}^*$ such that $(\mathcal{F}, \mathcal{F}^*)$ is an excellent pair (think of $\mathcal{F}^*$ as linear orders).

Theorem (Angel-Kechris-Lyons 2012)

Let $\mathcal{F}$ be a Hrushovski class that is part of an excellent pair $(\mathcal{F}, \mathcal{F}^*)$. Then $\mathrm{Aut}(\mathrm{Flim}(\mathcal{F}))$ is amenable.

What about Directed graphs?

Question

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Test Question 1

Is the class of tournaments (directed graphs where between each pair of nodes there is exactly one directed edge) a Hrushovski class?

Test Question 2

Is the class of (complete) bi-partite directed graphs a Hrushovski class?

Homogeneous directed graphs

Theorem, Cherlin 1998

There is an explicit classification of the classes of directed graphs that are Fraïssé classes.

Theorem, Jasiński-Laflamme-Nguyen Van Thé-Woodrow 2012

Every digraph on Cherlin's list is part of an excellent pair.

Question

Which of the classes on Cherlin's list are Hrushovski classes?

Cherlin's complete list of Fraïssé structure digraphs 1998

Structure	Name
$\mathbb{I}_\omega$	Independent set with countably many vertices
$\mathbb{Q}$	Rationals with directed edges agreeing with $\leq$
$\mathbb{T}^\omega$	Random countable tournament
$\mathbb{S}(2)$	Circle in two parts, rational arguments
$\mathbb{T}[\mathbb{I}_n]$	n-point blowup of a Fraïssé tournament
$\mathbb{I}_n[\mathbb{T}]$	"Fraïssé blowup" of an n-point independent set
$\hat{\mathbb{Q}}$	Two-point blowup of the rational digraph
$\hat{\mathbb{T}^\omega}$	Two-point blowup of the random countable tournament
$\mathbb{D}_n$	Random countable n-partite digraph
$\mathbb{S}$	The "Semigeneric" Digraph
$\mathbb{S}(3)$	Circle in three parts, rational arguments
$\mathbb{P}$	Random countable partial order
$\mathbb{P}(3)$	"Twisted" countable random partial order
$\mathbb{G}_n$	Random digraph without any n-point independent set
$\mathcal{F}(\mathcal{T})$	Random digraph without any finite tournaments from $\mathcal{T}$

What isn't Hrushovski?

Theorem

The following classes of directed graphs are not Hrushovski classes.

1. Linear Orders, partial orders - [Kechris-Sokic 2012].
2. Local linear orders $\mathbb{S}(2), \mathbb{S}(3)$ - [Kechris], [Zucker 2013].
3. 2-cover of $\mathbb{Q}$ and the "twisted" partial order $\mathbb{P}(3)$ - [P-Sokic 2015].
4. $\mathbb{Q}[\mathbb{I}_n], \mathbb{I}_n[\mathbb{Q}], \mathbb{S}(2)[\mathbb{I}_n]$, and $\mathbb{I}_n[\mathbb{S}(2)]$ - [P-Sokic 2015].

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Linear orders and partial orders: From the definition, use the invariant of "number of points above a given point".

Rest: Show non-amenability. Results are finite and "geometric".

New Hrushovski results

Theorem, P-Sabok 2015

The following classes of directed graphs are Hrushovski classes.

1. $\mathcal{T}$, the class of tournaments.
2. $\mathcal{D}_n$ with $n \leq \omega$, the class of complete n-partite digraphs.
3. $\mathcal{G}_n$ the class of digraphs that don't contain an independent set of size $n+1$.

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4. $\mathcal{F}(\mathcal{T})$, with $\mathcal{T}$ a collection of finite tournaments. The class of digraphs that don't contain any member of $\mathcal{T}$.
5. 2-cover of $\mathcal{T}$.
6. $\mathbb{S}$, the semigeneric digraph.

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Corollary

The Hrushovski status of each Fraïssé class of directed graphs is known.

General Strategy - Part 1

Let $\mathbb{A} \in \mathcal{K}$ and let $\phi_i : \mathbb{D}_i \to \mathbb{E}_i$ (for $i \in I$, with $N = |I|$) be a collection of isomorphisms where $\mathbb{D}_i, \mathbb{E}_i \leq \mathbb{A}$. Write $\mathbb{A} = (A, R_0^A, \ldots, R_{n-1}^A)$. Let F_N be the free group on generators $\{\phi_i : i \in I\}$.

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1. Describe an equivalence relation $\equiv$ on $A \times F_N$ from the isomorphisms $\{\phi_i\}_{i \in I}$.
2. Include all necessary relations on $X := {}^{A \times F_N} /_{\equiv}$.
3. Find a finite index subgroup $H \leq F_N$ such that $Y := {}^{X} /_{H}$ is well defined and embeds $\mathbb{A}$.
4. Include all remaining relations on Y that need to be included.

This construction is due to [Mackey 66].

General Strategy - Part 2

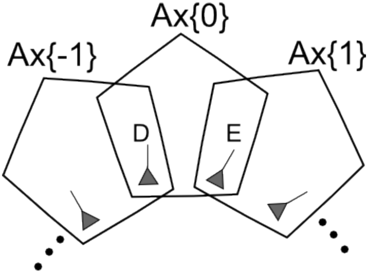

Figure : The case where there is only one isomorphism $\phi : \mathbb{D} \to \mathbb{E}$ (so $F_N = \mathbb{Z}$) and $\mathbb{D} \cap \mathbb{E} = \emptyset$.

General Strategy - Part 3

Let $\mathbb{A} \in \mathcal{K}$ and let $\phi_i : \mathbb{D}_i \to \mathbb{E}_i$ (for $i \in I$, with $N = |I|$) be a collection of isomorphisms where $\mathbb{D}_i, \mathbb{E}_i \leq \mathbb{A}$. Write $\mathbb{A} = (A, R_0^A, \ldots, R_{n-1}^A)$. Let F_N be the free group on generators $\{\phi_i : i \in I\}$.

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The equivalence relation

Consider the equivalence relation $\equiv$ on $A \times F_N$ where $(a, w) \equiv (b, v)$ if and only if $\exists r \geq 0$ such that $\phi_{i_k} \dots \phi_{i_1}(a) \in D_{i_k+1}$ for all $0 \leq k < r$, $\phi_{i_r} \dots \phi_{i_1}(a) = b$ and $\phi_{i_r} \dots \phi_{i_1} = wv^{-1}$ (or the same definition with a and b exchanged).

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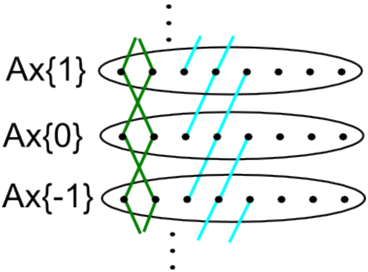
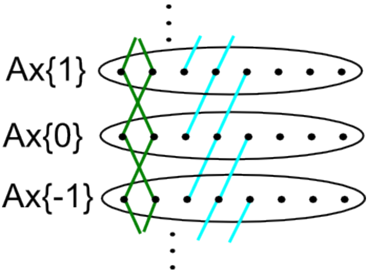


Figure : The case where there is only one isomorphism which exchanges a_1 and a_2, sends $a_5 \mapsto a_4$ and $a_4 \mapsto a_3$. The lines indicate the equivalence classes, and the edges are omitted.

Properties of the equivalence relation

Define $P_i : A \times F_N \to A \times F_N$, for $i \in I$ by the map $P_i(a, w) = (a, \phi_i w)$. Notice that

1. $(a, w) \equiv (b, w)$ iff $a = b$;
2. For $i \in I$, each P_i maps equivalence classes to equivalence classes (and is a bijection when restricted to any equivalence class).



Add the edges you absolutely have to.

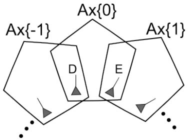
Let $X := {}^{A \times F_N}/_{\equiv}$ and let p_i be the map on X induced by P_i, for $i \in I$. In X, for $j < n$ consider the set

$$E_j := \{([a, 0], [b, 0]) : (a, b) \in R_j^A\}$$

which is well defined by (1) above. Also define

$$E_j' := \bigcup_{w \in F_N} w \cdot E_j,$$

where a word $w \in F_N$ acts on $(x, v) \in X$ by applying, in order, the appropriate p_i.



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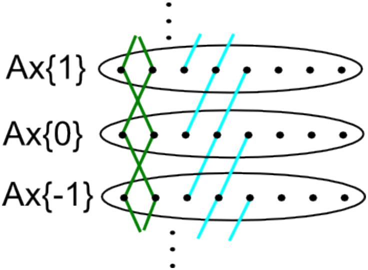
Goal now is to find an appropriate finite index subgroup $H \leq F_N$ so that $Y := {}^{X}/_{H}$ is well defined.



Glue things together

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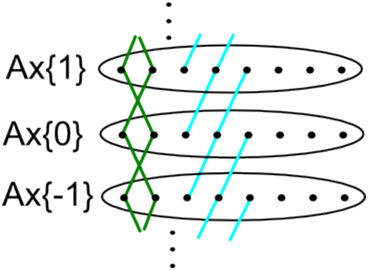

We need to make sure that points in $A \times \{0\}$ don't get identified.

Hall's Theorem

For each distinct pair $x, y \in A$ let $I(x, y) := \{v \in F_N : v \cdot x = y\}$, which is either empty or of the form $a(x, y) + H(x, y)$, for some $a(x, y) \in F_N$ and $H(x, y)$ a finitely generated subgroup of F_N.

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Hall's Theorem 1949

If G is a finitely generated free group, and H is a finitely generated subgroup of G, then H is closed in the profinite topology. In particular, if $H_1, \ldots, H_n$ is a collection of cosets of H, none of which contains the identity, then there is a finite index subgroup K that is disjoint from each coset H_i, for $1 \leq i \leq n$.

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By Hall's Theorem there is a finite index subgroup $H \leq F_N$ such that H is disjoint from all $I(x, y)$. Let $Y := {}^{X}/_{H}$.

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By Hall's Theorem there is a finite index subgroup $H \leq F_N$ such that H is disjoint from all $I(x, y)$. Let $Y := {}^{X}/_{H}$.
Let $h : \mathbb{A} \to Y$ be the isomorphism (!) $h(x) = [x, 0]$. For each $i \in I$, (1) p_i extends ϕ_i, and (2) p_i is an automorphism on Y.

Filling in the picture

Note that Y is a finite structure where every pair of vertices in Y has no relation, or a directed edge.

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Suppose that there is no relation between $[x, w]$ and $[y, v]$. So there is no relation between $z \cdot [x, w]$ and $z \cdot [y, v]$ for all $z \in F_N$.

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Suppose that there is no relation between $[x, w]$ and $[y, v]$. So there is no relation between $z \cdot [x, w]$ and $z \cdot [y, v]$ for all $z \in F_N$.
Each p_i remains an automorphism on this expanded Y and the map h remains an isomorphism. [End of Proof.]

Other classes - $\mathcal{D}_n$

The case of $\mathcal{D}_n$, the class of complete n-partite digraphs is similar, but we must take care to define the partition correctly.

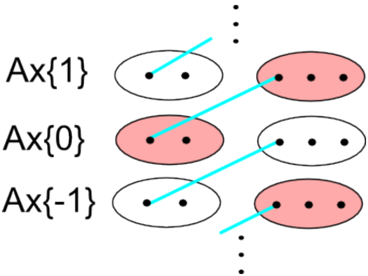

Figure : The case where $n = 2$ and there is only one isomorphism which exchanges $\perp$-equivalence classes. The dark classes form B_0 and the light classes form B_1.

Other classes - $\mathcal{A}(n, m)$

The class of finite **simple binary relational** structures $\mathcal{A}(n, m)$ proceeds exactly as the tournament case. This simulataneously captures the class of tournaments $\mathcal{T} = \mathcal{A}(1, 0)$ and the class of graphs $\mathcal{A}(2, 2)$, as well as many others.

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Definition

A binary relational structure $\mathbb{X}$ on a set X with binary relations $R_0, R_1, \ldots, R_{n-1}$ and $n \geq 2$ is *simple* if:

1. Each of the binary relations R_i is irreflexive.
2. There is a number $m < n$, the *symmetry number* of $\mathbb{X}$ so that:
   1. $R_i(x, y)$ implies $R_i(y, x)$ for all $x, y \in X$ and all $i \leq m$.
   2. $R_i(x, y)$ implies $\neg R_i(y, x)$ for all $x, y \in X$ and all $m \leq i < n$.
3. For all $i, j < n$ with $i \neq j$: $R_i(x, y)$ implies $\neg R_j(x, y)$ and $\neg R_j(y, x)$.
4. For all (x, y), there is an $i < n$ with $R_i(x, y)$.

Why this breaks down - Partial orders and cycles

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2. We can avoid cycles of length 10 and 70, but we will eventually have cycles.

1. However, this seems to work well when the classes are defined by "local" properties.

Thank You

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