Approximate Ramsey properties of matrices and finite dimensional normed spaces.

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Definition

1. *Let $\mathcal{M}_{n\times k}(\mathbb{F})$ be the set of all $n \times k$-matrices over $\mathbb{F}$.*
2. *Let $\mathcal{I}_{n\times k}(\mathbb{F})$ be the collection of $n \times k$-matrices of rank k.*
3. *Let $\mathrm{GL}_k(\mathbb{F})$ be the group of invertible $k \times k$-matrices.*
4. *Given a vector space V over $\mathbb{F}$, and $k \in \mathbb{N}$, let $\mathrm{Gr}(k, V)$ be the collection of all subspaces of V of dimension exactly k.*

Definition

Given $A \in \mathcal{I}_{n\times k}$, let

$$A = \mathrm{red}(A) \cdot \tau(A)$$

be the unique decomposition of A by the reduced column column form of A and a unique invertible matrix $\tau(A) \in \mathrm{GL}_k(\mathbb{F})$.
Let

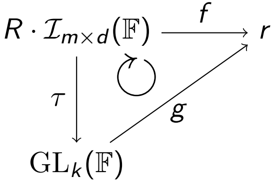
$$\mathcal{E}_{n\times k} := \{A \in \mathcal{I}_{n\times k} \,:\, \mathrm{red}(A) = A\}.$$

Theorem (Graham-Leeb-Rothschild; Kechris-Pestov-Todorcevic)

For every $k, m \in \mathbb{N}$ and every $r \in \mathbb{N}$ there exists n such that for every coloring

$$f : \mathcal{I}_{n\times k}(\mathbb{F}) \to r$$

there exists $R \in \mathcal{E}_{n\times m}$ and $g : \mathrm{GL}_k \to r$ such that



Since every subspace $W \in \mathrm{Gr}(k, \mathbb{F}^n)$ is the image of a matrix $A \in \mathcal{E}_{n\times k}$, the previous theorem gives the following.

Corollary

For every $k, m \in \mathbb{N}$ and $r \in \mathbb{N}$ there exists n such that for every coloring $c : \mathrm{Gr}(k, \mathbb{F}^n) \to r$ there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that c is constant on $\mathrm{Gr}(k, V)$.

GLR Theorem is a consequence of the Dual Ramsey Theorem.

Definition

Let $(S, <_S)$ and $(T, <_T)$ be two linearly ordered sets. A surjection $\theta : S \to T$ is called a rigid-surjection when $\min \theta^{-1}(t_0) < \min \theta^{-1}(t_1)$ for every $t_0 < t_1$ in T. Let $\mathrm{Epi}(S, T)$ be collection of all those surjections.

Theorem (Dual Ramsey Theorem; Graham and Rothschild)

For every finite linearly ordered sets S and T, and $r \in \mathbb{N}$ there exists $n \geq \#T$ such that every r-coloring of $\mathrm{Epi}(n, S)$ has a monochromatic set of the form $\mathrm{Epi}(T, S) \circ \sigma$ for some $\sigma \in \mathrm{Epi}(n, T)$.

Sketch of Proof of GLR

Definition

We order first $\mathbb{F}$ in a way that 0 and 1 are the first two elements, and then each power $\mathbb{F}^k$ antilexicographically. Let $\Phi : \mathrm{Epi}(n, \mathbb{F}^k) \to \mathcal{I}_{n\times k}$ be defined for $\sigma \in \mathrm{Epi}(n, \mathbb{F}^k)$ as the matrix $A = \Phi(\sigma)$ whose ξ-row vector is $\sigma(\xi)$, for each $\xi < n$. $\Phi(\sigma)$ is the σ-Matrix.

For fixed integers k, m and r, let n be the Graham-Rothschild number for $\mathbb{F}^k$, $\mathbb{F}^m$, and number of colors $r^{\mathrm{GL}_k(\mathbb{F})}$. We claim that n works.
Fix $f : \mathcal{I}_{n\times m} \to r$. Let $c : \mathrm{Epi}(n, \mathbb{F}^k) \to r^{\mathrm{GL}_k(\mathbb{F})}$ be the mapping

$$\begin{aligned} \sigma \mapsto c(\sigma) : \mathrm{GL}_k(\mathbb{F}) &\to r \\ B &\mapsto f(\Phi(\sigma)\cdot B) \end{aligned}$$

Let $\varrho \in \mathrm{Epi}(n, \mathbb{F}^m)$ be such that c is constant on $\mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k) \circ \varrho$ with value $g \in r^{\mathrm{GL}_k(\mathbb{F})}$.

Sketch of Proof of GLR

Let R be the ϱ-matrix

$$R := \Phi(\varrho).$$

Claim

For $A \in \mathcal{I}_{m\times k}$ one has that

$$f(R\cdot A) = g(\tau(R\cdot A)) = g(\tau(A)).$$

Fix such A; decompose it $A = \mathrm{red}(A)\cdot\tau(A)$. Then

$$R\cdot A = \Phi(\varrho)\cdot\mathrm{red}(A)\cdot\tau(A).$$

It is easy to see that

$$\mathrm{red}(A)^{\mathrm{t}} \in \mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k).$$

It is also easy that

$$\Phi(\varrho)\cdot\mathrm{Red}(A) = \Phi(\mathrm{Red}(A)^{\mathrm{t}}\circ\varrho).$$

Sketch of Proof of GLR

Hence, setting $\sigma := \mathrm{Red}(A)^{\mathrm{t}}$,

$$f(R \cdot A) = f(\Phi(\sigma \circ \varrho) \cdot \tau(A)) = c(\sigma \circ \varrho)(\tau(A)) = g(\tau(A)).$$

It is not difficult to see the minimality of $\tau : \mathcal{I}_{n\times k} \to \mathrm{GL}_k(\mathbb{F})$; that is,

given any $R \in \mathcal{I}_{n\times m}$ and any $B \in \mathrm{GL}_k(\mathbb{F})$ there is $A \in \mathcal{I}_{m\times k}$ such that $\tau(R \cdot A) = B$.

Definition

Given k, n, we consider $\mathcal{M}_{n\times k}(\mathbb{F})$ as a metric space by considering the matrix norm

$$\|A\|_\infty := \max_{\|v\|_\infty = 1} \|Av\|_\infty$$

Definition

Given an integer k, let $\mathcal{N}_k$ be the collection of all norms $M : \mathbb{F}^k \to \mathbb{R}$.

Example

Given $v = (v_i)_{i<n} \in \mathbb{F}^n$,

$$N_\infty(v) = \|v\|_\infty := \max_{i<n} |v_i|$$

$$N_p(v) = \|v\|_p := \left(\sum_{i<n} |v_i|^p \right)^{\frac{1}{p}}, \; p \geq 1.$$

Definition

We say that $M \leq N$ *if* $M(v) \leq N(v)$ *for every* $v \in \mathbb{F}^k$. *Let*

$$[M, N] := \{P \in \mathbb{F}^d \, : \, M \leq P \leq N\}$$

Observe that $\bigcup_{0<a\leq b<\infty}[aM, bM]$ for every norm $M \in \mathcal{N}_k$.

$\mathcal{N}_k$ has natural metrics:

Definition

Given $M, N \in \mathcal{N}_k$, let

$$d_0(M,N) := d_{\mathcal{H}}^{\ell_1}(B_{(\mathbb{F}^k, M^*)}, B_{(\mathbb{F}^k, N^*)})$$
$$d_1(M,N) := \log \max\{\|Id\|_{M,N}, \|Id\|_{N,M}\},$$

where

(a) *$d_{\mathcal{H}}^{\ell_1}(K,L)$ is the Hausdorff distance between two compacta $K, L \subseteq \mathbb{F}^k$ when considering $\mathbb{F}^k$ endowed with the ℓ_1-metric.*

(b) *M^* is the dual norm of M, defined by*

$$M^*(v) := \max_{M(w)=1} w^* v$$

and $B_{(\mathbb{F}^k, P)} := \{v \in \mathbb{F}^k : P(v) \leq 1\}$ is the unit ball of $(\mathbb{F}^k, P)$.

Remark

1. *$\mathcal{N}_k$ is not compact.*
2. *If $aN_\infty \le M_0, M_1 \le bN_\infty$ then*

$$ad_0(M_0, M_1) \le d_1(M_0, M_1) \le bd_0(M_0, M_1).$$

Definition

Let $\mathcal{I}_{n\times k}$ be the set of $n \times k$-matrices of rank k. Let

$$\tau_\infty : \mathcal{I}_{n\times k} \to \mathcal{N}_k$$

be defined by $\tau_\infty(A)(v) := N_\infty(Av)$. Let $\mathcal{E}_{n\times k}$ be the set of all matrices $A \in \mathcal{I}_{n\times k}$ such that $\tau_\infty(A) = N_\infty$; that is, A defines an isometry between ℓ_∞^k and ℓ_∞^n.

Proposition

$\tau_\infty : (\mathcal{I}_{n\times k}, N_\infty) \to (\mathcal{N}_k, d_0)$ is 1-Lipschitz, and

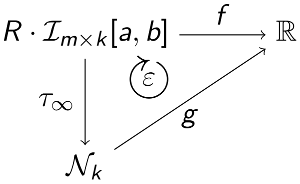
$$\bigcup_{n\geq k} \tau_\infty(\mathcal{I}_{n\times k}) \text{ is dense in } \mathcal{N}_k.$$

Theorem

For every k, m, $0 < a \leq b$ and every $\varepsilon > 0, L > 0$ there exists n such that for every Lipschitz coloring

$$f : \mathcal{I}_{n\times k} \to \mathbb{R}$$

with $\mathrm{Lip}(f) \leq L$ there exists $R \in \mathcal{E}_{n\times m}$ and $g : \mathcal{N}_k \to \mathbb{R}$ such that $\mathrm{Lip}(g) \leq (1+\varepsilon)\mathrm{Lip}(f)$ and



Remark

The previous result is true for a large class of metric spaces, not only $\mathbb{R}$*; in particular for* $\ell_\infty^n(\mathbb{R})$*.*
$\tau_\infty : \mathcal{I}_{n\times k} \to \mathcal{N}_k$ *is unique in some precise sense.*

Definition

Given $V, W \in \mathrm{Gr}(k, \mathbb{F}^n)$, let $d(V, W)$ be the Hausdorff distance between $B_{(V,N_\infty)}$ and $B_{(W,N_\infty)}$ with respect to the ∞-distance on $\mathbb{F}^n$.

Definition

Let

$$d_{\mathrm{BM}}(M, N) := \log \inf_{A \in \mathrm{GL}_k} \|A\|_{M,N} \cdot \|A^{-1}\|_{N,M}$$

be the Banach-Mazur pseudo metric on $\mathcal{N}_K$. Let

$$\mathcal{B}_k := \mathcal{N}_k / \{d = 0\}.$$

$\mathcal{B}_k$ is the k-Banach-Mazur compactum(which is compact)

Definition

Let $\nu_\infty : \mathrm{Gr}(k, \mathbb{F}^n) \to \mathcal{B}_k$ *be defined for* $V \in \mathrm{Gr}(k, \mathbb{F}^n)$ *as the isometry class of the normed space* (V, N_∞)*, i.e.,*

$$\nu_\infty(V) = [\tau_\infty(A)]_{\mathrm{BM}}$$

where $A \in \mathcal{I}_{n \times k}$ *is such that* $\mathrm{Im} A = V$*.*

Proposition

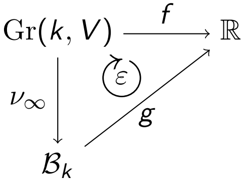
ν_∞ *is* k^2*-Lipschitz.*

Theorem

For every k, m, and every $\varepsilon > 0, L > 0$ there exists n such that for every Lipschitz coloring

$$f : \mathrm{Gr}(k, \mathbb{F}^n) \to \mathbb{R}$$

with $\mathrm{Lip}(f) \leq L$ there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ with (V, N_∞) linearly isometric to ℓ_∞^m and $g : \mathrm{Gr}(k, V) \to \mathbb{R}$ such that $\mathrm{Lip}(g) \leq (1+\varepsilon)\mathrm{Lip}(f)$ and



Since the ℓ_∞^n's are almost-universal, for large N, ℓ_∞^N has an almost-isometrical V copy of ℓ_2^m, that is, (V, N_∞) is almost isometric to ℓ_2^m. Hence, $\nu_\infty(\mathrm{Gr}(k, V))$ as a small diameter. It readily follows the following.

Corollary

For every k, m, and every $\varepsilon > 0, L > 0$ there exists n such that for every Lipschitz coloring

$$f : \mathrm{Gr}(k, \mathbb{F}^n) \to \mathbb{R}$$

with $\mathrm{Lip}(f) \leq L$ there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that

$$\mathrm{osc}(f \restriction \mathrm{Gr}(k, V)) < \varepsilon.$$

Definition

We say that a collection of Banach spaces $\mathcal{F}$ has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$, $K > 0$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that $\mathrm{Emb}(G, H) \neq \emptyset$ and such that for every Lipschitz map $f : \mathrm{Emb}(F, H) \to \mathbb{R}$ with $\mathrm{Lip}(f) \leq L$ there exists

$$\varrho \in \mathrm{Emb}(G, H).$$

such that

$$\mathrm{osc}(f \restriction \varrho \circ \mathrm{Emb}(F, G)) < \varepsilon.$$

$(\mathbb{F} = \mathbb{R})$

Definition

Recall that a finite dimensional space F is called polyhedral when its unit ball is a polyhedron; that is, it has finitely many extreme points.

It is well-known that a f.d. space is polyhedral if and only if can be isometrically embedded into some ℓ_∞^n. Polyhedral spaces are dense in the class of finite dimensional spaces. Polyhedral spaces are isometric to $(\mathbb{F}^k, \tau_\infty(A))$ for some k and $A \in \mathcal{I}_{n\times k}$.

Theorem

The Polyhedral spaces have the ARP.

Proof.

This is a particular case of our result over matrices. □

However our proof of the ARP of injective matrices is a consequence of the previous Theorem. In fact, we prove first the following particular case:

Theorem

The class of isometric spaces to some ℓ_∞^n have the ARP.

The following is used to prove the ARP of polyhedral spaces.

Proposition

Every polyhedral space F has an injective envelope. That is, there is some n and an isometric embedding $T_F : F \to \ell_\infty^n$ such that for any other isometric embedding $U : F \to \ell_\infty^k$ there is an isometric embedding $\Theta : \ell_\infty^n \to \ell_\infty^k$ such that $U = \Theta \circ T_F$.

Theorem

The class of f.d. normed spaces have the ARP.

Polyhedral spaces are dense in the class of f.d. spaces. So, an isometric embedding T between two f.d. spaces X and Y will induce a θ-embedding T' ($\theta^{-1}\|x\| \le \|T'x\| \le \theta\|x\|$) between polyhedral spaces X' and Y' appropriately closed to X and Y. Our previous ARP is about isometric embeddings between polyhedral spaces. So, somehow, we have to turn T' into an isometric embedding between X' and Y'. In general, it is not true that a θ-isometric embedding is θ'-close to an isometric embeddings, as there are many spaces with two isometries, yet with many approximate isometries. We use following extension of a result by Kubis and Solecki.

Proposition

Let $(X_i)_{i\le n}$ be f.d. spaces, and $1<\theta<\tau$. Then there is a f.d. space Y having isometric copies of each X_i and an isometric embedding $J : X_n \to Y$ such that for every θ-embedding $T : X_i \to X_n$ there is an isometric embedding $I : X_i \to Y$ such that $\|I - J \circ T\| < \tau - 1$.

Corollary

The group of (linear) isometries $\mathrm{Iso}(\mathbb{G})$ of the Gurarij space $\mathbb{G}$ (the metric Fraïssé limit of f.d. normed spaces), with the pointwise topology is extremely amenable.

Corollary

The universal minimal flow of the group of affine homeomorphism of the Poulsen simplex (the metric inverse Fraïssé limit of f.d. simplexes) with the uniform topology is the Poulsen simplex with its natural action.

Corollary

The finite metric spaces have the ARP.

Definition

Let $(S, <_S)$ and $(T, <_T)$ be two linearly ordered sets. A surjection $\theta : S \to T$ is called a min-*surjection* *when $\min \theta^{-1}(t_0) < \min \theta^{-1}(t_1)$ for every $t_0 < t_1$ in T. Let $\mathrm{Epi}(S, T)$ be collection of all those surjections.*

Theorem (Dual Ramsey Theorem; Graham and Rothschild)

For every finite linearly ordered sets S and T, and $r \in \mathbb{N}$ there exists $n \geq \#T$ such that every r-coloring of $\mathrm{Epi}(n, S)$ has a monochromatic set of the form $\mathrm{Epi}(T, S) \circ \sigma$ for some $\sigma \in \mathrm{Epi}(n, T)$.

Definition

Let $\mathcal{E}_{n\times k}$ be the collection of all $n \times k$ matrices representing (in the unit bases of $\mathbb{F}^k$ ad $\mathbb{F}^n$) a linear isometry between ℓ_∞^k and ℓ_∞^n.

Proposition

$A \in \mathcal{E}_{n\times k}$ if and only if each column vector has ∞-norm one and each row vector has ℓ_1-norm at most 1.

Given $\varepsilon > 0$, let $\mathcal{N}$ be a finite ε-dense subset of the unit ball $B_{\ell_1^k}$

1. containing 0 and the unit vectors u_i, and
2. such that for every non-zero $v \in B_{\ell_1^k}$ there is $w \in \mathcal{N}$ such that $\|v - w\|_1 < \varepsilon$ and $\|w\|_1 < \|v\|_1$. e.g., for large $l\varepsilon \geq 1$,

$$\mathcal{N} = (\{\pm \frac{i}{kl}\}_{i \leq kl})^k \cap B_{\ell_1^k}$$

Let $<$ be any total ordering on $\mathcal{N}$ such that $v < w$ when $\|v\|_1 < \|w\|_1$. We order n canonically.

Definition

Let $\Phi : \mathrm{Epi}(n, \mathcal{N}) \to \mathcal{E}_{n\times k}$ *be defined for a* $\sigma : n \to \mathcal{N}$ *as the* $n \times k$*-matrix* A_σ *whose* ξ*-row vector,* $\xi < n$*, is* $\sigma(\xi)$*.*

It is easy to see that $\Phi(\sigma) \in \mathcal{E}_{n\times k}$. To simplify, suppose that $\mathbb{F} = \mathbb{R}$.

Proposition

There is a finite set $\Gamma \subseteq \mathcal{E}_{n\times k}$ *such that for every other* $A \in \mathcal{E}_{n\times k}$ *there exists* $B \in \Gamma$ *such that*

$$A^{\mathrm{t}} B = \mathrm{Id}_k.$$

We order now $\Delta := \mathcal{N} \times \Gamma$ lexicographically, where Δ is arbitrarily ordered. Given now k, m, a number of colors r, we use apply the DR theorem to $\mathcal{N}$ and Δ to find the corresponding n. Then n works: Given $c : \mathcal{E}_{n\times k} \to r$, we have the induced color

$$c \circ \Phi : \mathrm{Epi}(n, \mathcal{N}) \to r$$

Let $\varrho \in \mathrm{Epi}(n, \Delta)$ such that c is constant on $\mathrm{Epi}(\Delta, \mathcal{N}) \circ \varrho$. Let now $R \in \mathcal{E}_{n\times m}$ be the matrix whose ξ-column is Av where $\varrho(\xi) = (v, A)$.

Proposition

For every $B \in \mathcal{E}_{m\times d}$ there exists $\sigma \in \mathrm{Epi}(\Delta, \mathcal{N})$ such that $\|RB - \Phi(\varrho \circ \sigma)\|_\infty < \varepsilon$.

1. There are similar results for other $p's$, i.e. for f.d. subspaces of $L_p[0,1]$. (Joint work with V. Ferenczi, B. Mbombo and S. Todorcevic)
2. There is an extension to non-commutative spaces (operator spaces and systems). (Joint work with M. Lupini).