Structural Ramsey theory in the projective setting

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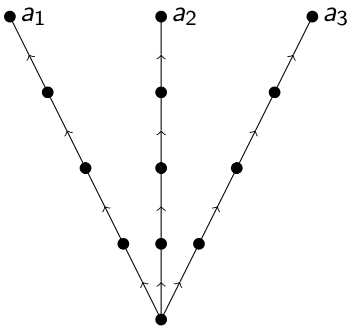
Structures (A, R^A, S^A), the directed graph (A, R^A) looks as below, for some ordering $a_1 < a_2 < \ldots < a_n$ of branches of A we have $S^A(x, y)$ if and only if there are $i \leq j$ such that $x \in a_i$ and $y \in a_j$.



Ramsey classes

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- Let $\mathcal{G}$ be a family of finite structures.
- For $A, B \in \mathcal{G}$, denote by $\binom{B}{A}$ the set of all epimorphisms from B onto A.
- Say that $\mathcal{G}$ is a *Ramsey class* if for every $A, B \in \mathcal{G}$ and a natural number $r \geq 2$ there exists $C \in \mathcal{G}$ such that for every colouring c of $\binom{C}{A}$ with r colours there exists $g \in \binom{C}{B}$ such that $\binom{B}{A} \circ g = \{f \circ g : f \in \binom{B}{A}\}$ is c-monochromatic, that is, c restricted to $\binom{B}{A} \circ g$ is constant.

Ramsey theorem

Theorem

The class $\mathcal{F}_<$ is a Ramsey class.

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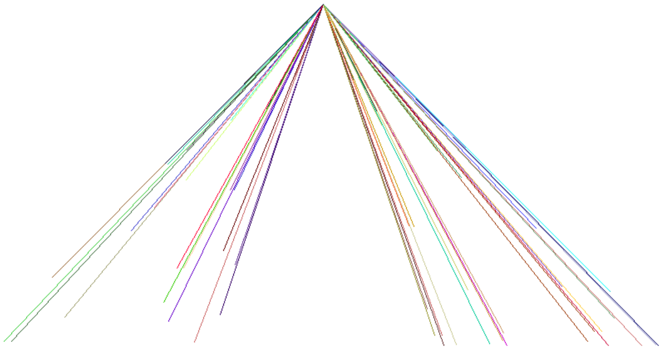
3. The group $H(L_<)$ of homeomorphisms of the ordered Lelek fan is extremely amenable.

Explanation

What is $H(L_<)$?

The Lelek fan L



Size-insensitivity 1

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For $B_i \subset N$ for $i = 1, 2 \ldots, m$, we say that c is size-insensitive on $(B_i)_{i=1}^{m}$ if whenever $A_i, A'_i \subset B_i$ with $0 \leq |A_i| = |A'_i| \leq k$ for $i = 1, 2, \ldots, m$ then

$$c(A_1, \ldots, A_m) = c(A'_1, \ldots, A'_m).$$

Size-insensitivity 2

Theorem

Let m, k, l be natural numbers such that $k \leq l$. Then there exists N such that for every colouring

$$c : \prod_{i=1}^{m} \bigcup_{j=0}^{k} N^{[j]} \to \{1, 2, \ldots, r\}$$

there exist $B_1, \ldots, B_m \subset N$ with $|B_i| = l$ such that c is size-insensitive on $(B_i)_{i=1}^{m}$.

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$$\mathrm{supp}(p) = \{l \in \{1, \ldots, n\} \colon p(l) \neq 0\}$$

OPERATIONS

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Operations on $\mathrm{FIN}_k(n)$

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- $+$ $p + q$ is defined to be $p \cup q$, whenever $\max(\mathrm{supp}(p)) < \min(\mathrm{supp}(q))$
- $T_i^{(k)}$ For every $i = 1, 2, \ldots, k$ we have a function $T_i^{(k)} : \mathrm{FIN}_k(n) \to \mathrm{FIN}_{k-1}(n)$

$$T_i^{(k)}(p)(l) = \begin{cases} p(l) & \text{if } p(l) < i \\ p(l) - 1 & \text{if } p(l) \geq i. \end{cases}$$

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Let also $T_0^{(k)} = \mathrm{id} \restriction_{\mathrm{FIN}_k}$.

Generalization of the finite version of the Gowers' Ramsey Theorem, part 1

A sequence $B = (b_s)_{s=1}^m$ of elements of $\mathrm{FIN}_k(n)$ is called a block sequence if for every s,

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Let $\langle B \rangle_k$ denote the subset of $\mathrm{FIN}_k(n)$ of

$$\sum_{s=1}^m T_{\vec{i_s}}(b_s),$$

such that $\vec{i_s} \in \prod_{j=1}^k \{0, 1, \ldots, j\}$ and there is an s with $T_{\vec{i_s}}(b_s) = b_s$.

Generalization of the finite version of the Gowers' Ramsey Theorem, part 2

Theorem

Let k, m, r be natural numbers. Then there exists a natural number n such that for every colouring $c\colon \mathrm{FIN}_k(n) \to \{1, 2, \ldots, r\}$ there exists a block sequence B of length m in $\mathrm{FIN}_k(n)$ such that $\langle B \rangle_k$ is c-monochromatic.

Generalization of the finite version of the Gowers' Ramsey Theorem, part 3

We proved and need a more general Ramsey theorem.

Theorem

Let $k, l \geq k, m, m \geq d, r$ be natural numbers. Then there exists a natural number n such that for every colouring $c \colon \mathrm{FIN}_k^{[d]}(n) \to \{1, 2, \ldots, r\}$ there exists a block sequence B of length m in $\mathrm{FIN}_l(n)$ such that $\langle B \rangle_{k,l}^{[d]}$ is c-monochromatic.

Remarks on the Gowers' Ramsey Theorem

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- The Gowers' Ramsey Theorem involves only operations T_1 and $+$.
- The Gowers' Ramsey Theorem is true for finite and also for countable infinite block sequences.

About the proof 1

For a block sequence in $A = (a_i)_{i=1}^m$ in $\mathrm{FIN}_1(n)$ let

$$\mathrm{FIN}_k(A) = \{\sum_{j=1}^m i_j \cdot \chi(a_j) : i_j = 0, 1, \ldots, k \text{ and for some } j \text{ we have } i_j = k\}.$$

About the proof 2 – type insensitivity

Lemma (Tyros)

For every triple (k, m, r) *of natural numbers, there exists* n *such that for every colouring* $c : \mathrm{FIN}_k(n) \to \{1, 2, \ldots, r\}$, *there is a block sequence* A *of length* m *in* $\mathrm{FIN}_1(n)$ *such that any two elements in* $\mathrm{FIN}_k(A)$ *of the same type have the same colour.*

About the proof 3

The block sequence - solution is a sequence of m pyramids $B = (b_i)_{i=1}^{m}$ over a type insensitive block sequence $A = (a_i)_{i=1}^{m(2k-1)}$ in $\mathrm{FIN}_1(n)$.

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The i-th pyramid is:

$$b_i = \sum_{j=-(k-1)}^{k-1} (k - |j|) \cdot \chi(a_{q_i + j})$$

where $q_i = (i-1)(2k-1) + k$.