Connected-homogeneous digraphs

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Banff, 12 November 2015

Open problem

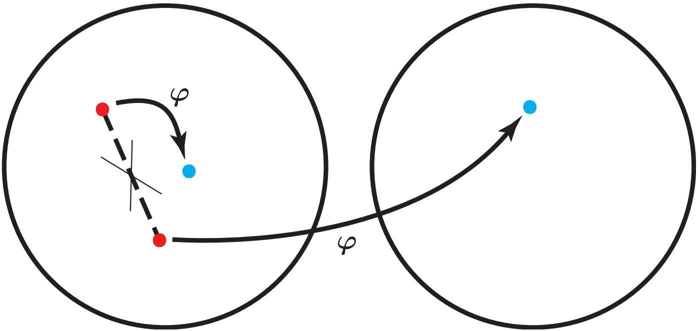
Problem

Classify the infinite one-ended C-homogeneous digraphs that are not locally finite.

The problem is solved

Done!

If homogeneous graphs are not connected



How to overcome this restriction

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Definition

A graph is connected-homogeneous, or C-homogeneous, if every isomorphism between any two finite connected induced subgraphs extends to an automorphism of the whole graph.

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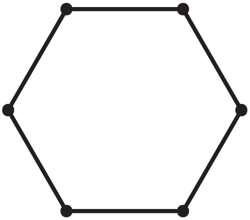

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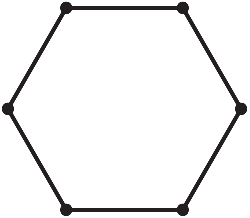

- C-homogeneous (di-)graphs need not be homogeneous.

What is known?

	homogeneous	C-homogeneous
partial orders	Schmerl (1979)	Gray&Macpherson (2010)
graphs	Gardiner (1976) Lachlan&Woodrow (1980)	Gardiner (1978) Enomoto (1981) Gray&Macpherson (2010)
digraphs	Lachlan (1982) Cherlin (1998)	Gray&Möller (2011) H&Hundertmark (2012) H (2015^{+}&2015^{++})

Strategy of the proof

Question

How can we obtain structural facts from the property 'C-homogeneous' that will help us in the proof?

First structural fact

Lemma

1. *The out-neighbourhood of some (and hence every) vertex of a C-homogeneous digraphs induces a homogeneous digraph.*
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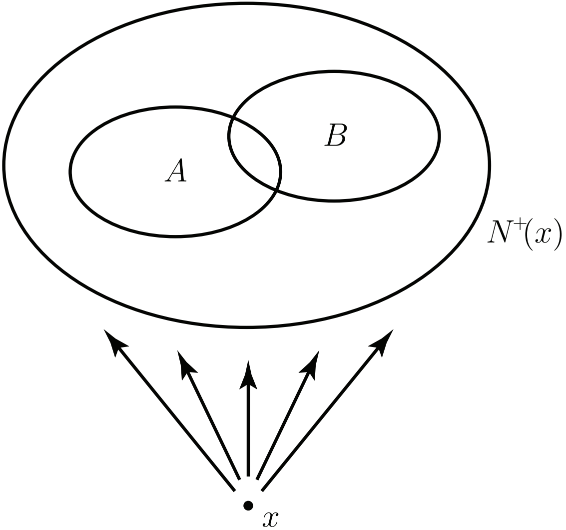
First structural fact (proof)



1^{st} result

Theorem 1

For every countable C-homogeneous digraph D one of the following statements is true:

1. *D is a blow-up of a homogeneous digraph;*
2. *D has more than one end;*
3. *every vertex of D has an independent out- and an independent in-neighbourhood.*

2^{nd} result

Theorem (Dunwoody&Krön 2015)

For transitive graphs G with more than one end there is an Aut(G)*-invariant nested set of vertex cuts distinguishing some ends.*

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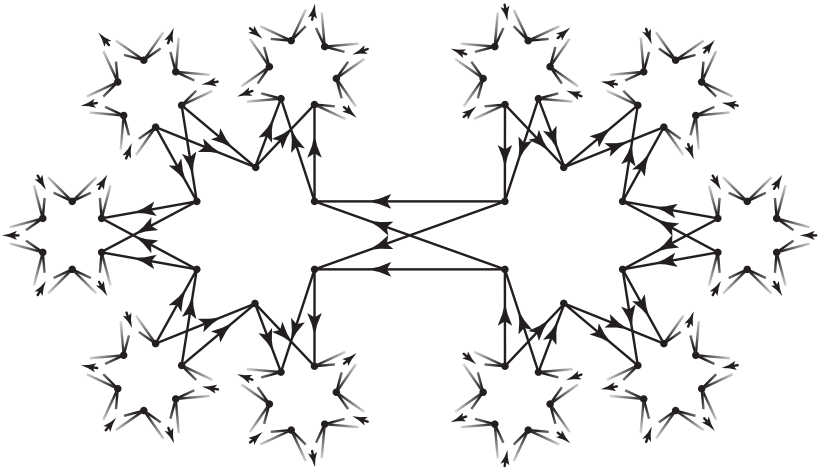
Theorem 2

Connected C-homogeneous digraphs with at least two ends have connectivity 1 *or* 2 *and are tree-like.*

There are five classes of such digraphs.

An infinitely ended C-homogeneous digraph



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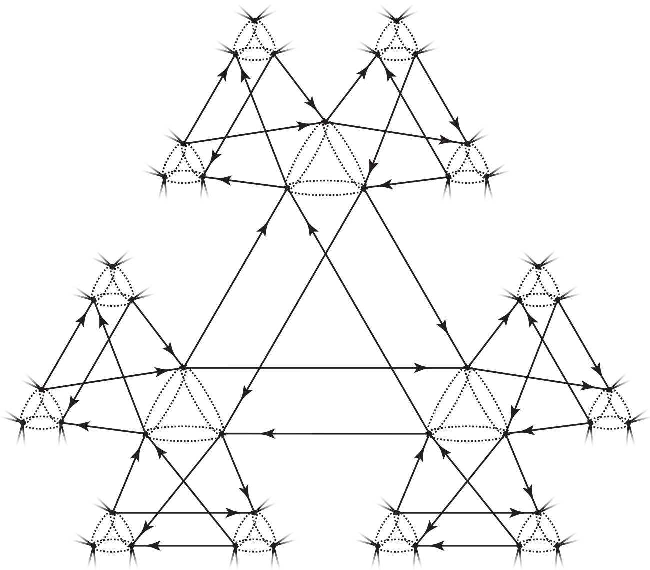
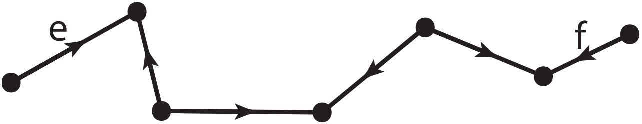


Reachability

Definition

An edge e is reachable from an edge f if there is some walk $x_1 \dots x_n$ containing e and f such that:

$$x_{i-1} \in N^+(x_i) \Leftrightarrow x_{i+1} \in N^-(x_i).$$



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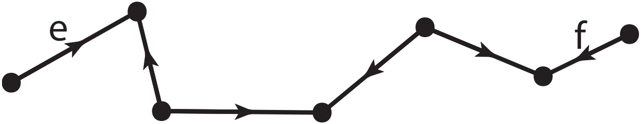

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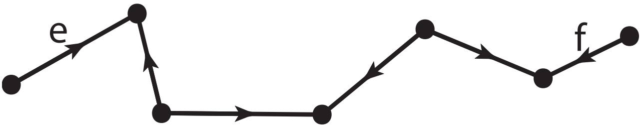

Remark

Reachability is an equivalence relation.

Lemma (Cameron&Praeger&Wormald 1993)

In edge-transitive digraphs either the reachability relation is universal or one (and hence every) equivalence class forms a bipartite digraph.

Reachability

Lemma (Gray&Möller 2011)

In C-homogeneous digraphs whose reachability relation is not universal and with independent out- and in-neighbourhood for every vertex, the equivalence classes of the reachability relation form (non-empty) C-homogeneous bipartite digraphs.

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All C-homogeneous bipartite digraphs can be easily obtained from the homogeneous bipartite graphs.

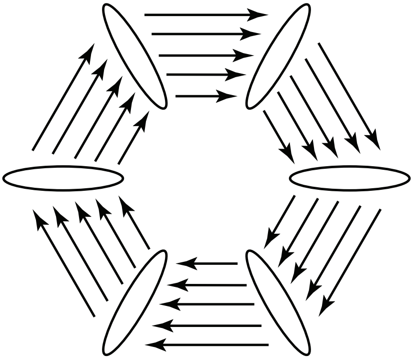
3^{rd} result

Theorem 3

For every countable C-homogeneous digraph D with at most end whose reachability relation is not universal and with independent out- and in-neighbourhood for every vertex one of the following statements is true.

1. *D is a (random) blow-up of directed cycles or the double ray.*
2. *D is the generic 2-partite digraph.*
3. *D is a quotient digraph of D^*.*

Blow-ups

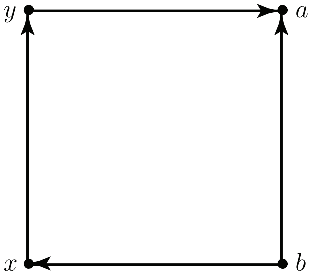


The bipartite digraphs are either

- complete or
- complements of perfect matchings or
- generic bipartite.

Universal reachability relation

If the reachability relation is universal, then the digraph contains the following *induced* subdigraph:



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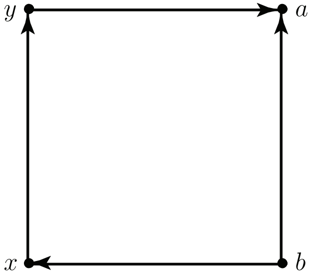

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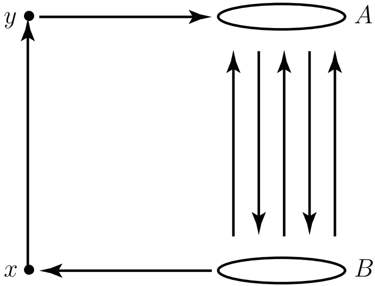

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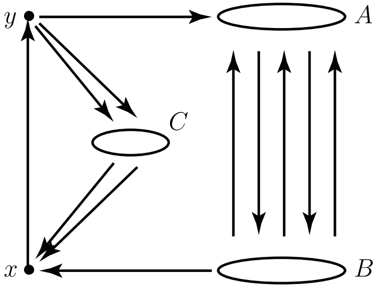
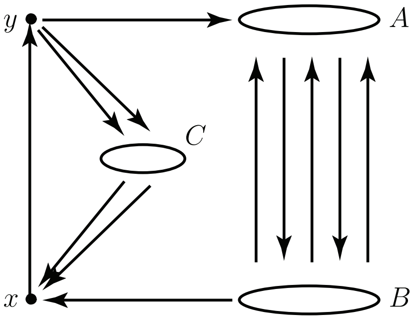


Universal reachability relation

Lemma

If D is a countable C-homogeneous digraph with universal reachability relation and with independent out- and in-neighbourhood for every vertex, then with
$A := N^+(y) \smallsetminus N^-(x)$ *and*
$B := N^-(x) \smallsetminus N^+(y)$
for $xy \in E(D)$ *the digraph induced by* $A \cup B$ *is a non-empty homogeneous 2-partite digraph.*



4^{th} result

Theorem 4

A countable C-homogeneous digraph with universal reachability relation and with independent out- and in-neighbourhood for every vertex is homogeneous.

Main ingredients of the proof

Our classification needed the classification of the countable homogeneous

- digraphs
  (Cherlin 1998)
- bipartite graphs
  (Goldstern&Grossberg&Kojman 1996)
- 2-partite digraphs
  (H 2014)

The classification

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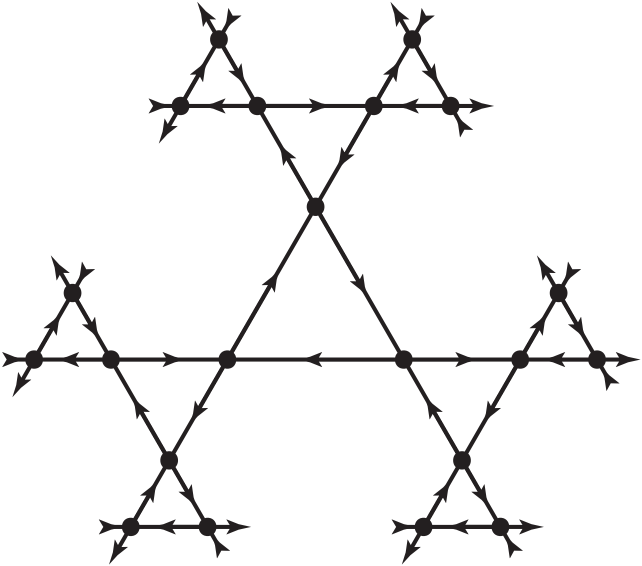
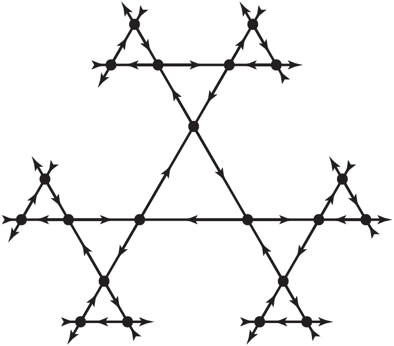
- Eleven of these classes have explicit constructions but
- one does not!

A C-homogeneous digraph: D^*



One particular class



A special class of C-homogeneous digraph of degree 4:
D^* factorised by some non-universal $\mathrm{Aut}(D^*)$-invariant equivalence relation

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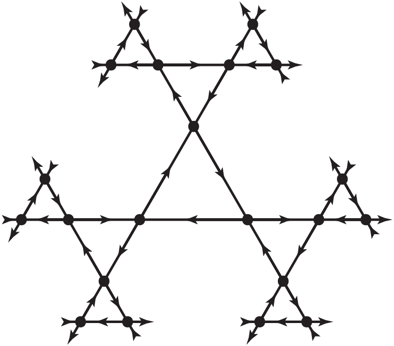

A special class of C-homogeneous digraph of degree 4:
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Theorem

There is a canonical bijection from this class of C-homogeneous digraphs to those subgroups of the modular group $C_2 * C_3$ that contain a fixed involution.

The classification

Theorem

A countable digraph is C-homogeneous if and only if it is a disjoint union of countably many copies of one of the following digraphs:

(i) *a countable homogeneous digraph;*

(ii) $H[I_n]$ *for some* $n \in \mathbb{N}^\infty$ *and with either* $H = S(3)$ *or* $H = T^\wedge$ *for some countable homogeneous tournament* $T \neq S(2)$*;*

(iii) $X_\lambda(T)$ *for some countable homogeneous tournament* T *and* $\lambda \in \mathbb{N}^\infty$*;*

(iv) *a regular tree;*

(v) $DL(\Delta)$, *where* Δ *is a bipartite digraph such that* $G(\Delta)$ *is one of*

- C_{2m} *for some integer* $m \geq 2$,
- CP_k *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$,
- $K_{k,l}$ *for* $k, \ell \in \mathbb{N}^\infty$, $k, \ell \geq 2$, *or*
- *the countable generic bipartite graph;*

(vi) $M(k, m)$ *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$ *and some integer* $m \geq 2$*;*

(vii) $M'(2m)$ *for some integer* $m \geq 2$*;*

(viii) Y_k *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$*;*

(ix) $C_m[I_k]$ *for some* $k, m \in \mathbb{N}^\infty$ *with* $m \geq 3$*;*

(x) $\mathcal{R}_m$ *for some* $m \in \mathbb{N}^\infty$ *with* $m \geq 3$*;*

(xi) *the generic 2-partite digraph; or*

(xii) $X_2(C_3)_\sim$, *where* $\sim$ *is a non-universal* $Aut(X_2(C_3))$*-invariant equivalence relation on* $V(X_2(C_3))$.