Set-homogeneous structures

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Set-homogeneity

Definition

A relational structure M is set-homogeneous if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $Ug = V$.

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Comparison with homogeneity...

- Homogeneity: whenever $U \cong V$ then every isomorphism between U and V extends to an automorphism.
- Set-homogeneity: whenever $U \cong V$ then some isomorphism between U and V extends to an automorphism.

Clearly if M is homogeneous then M is set-homogeneous.

General question

How much stronger is homogeneity than set-homogeneity?

Set-homogeneity

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Aim: To tell you everything I know about finite and countably infinite set-homogeneous structures. The majority of the results I will talk about come from the following two papers:

M. Droste, M. Giraudet, D. Macpherson, N. Sauer,
Set-homogeneous graphs.
J. Combin. Theory Ser. B 62 (1994) 63–95.

R. D. Gray, D. Macpherson, C. E. Praeger, G. F. Royle
Set-homogeneous directed graphs.
J. Combin. Theory Ser. B 102 (2012) 474–520.

There are also a number of nice open problems in this area that I also want to draw your attention to.

Set-homogeneity vs homogeneity

- Clearly in general: homogeneous $\Rightarrow$ set-homogeneous.
- The converse is not true in general.

Example

Let $M = (\mathbb{Q}, R)$ where R is the ternary relation given by:

$$\forall x, y, z \in M,\ (x, y, z) \in R \Leftrightarrow x < y < z.$$

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$$\forall x, y, z \in M,\ (x, y, z) \in R \Leftrightarrow x < y < z.$$

- M is set-homogeneous
  - any order-preserving bijection between between finite substructures is an isomorphism that extends to an automorphism
- M is not homogeneous
  - $(0, 1) \mapsto (1, 0)$ is an isomorphism between substructures
  - it does not extend to an automorphism since $(0, \frac{1}{2}, 1) \in R$ but $(1, x, 0) \notin R$ for any $x \in \mathbb{Q}$.

Some history

- Set-homogeneity is a concept originally due to Fraïssé and Pouzet.
- In a finite relational language, any set-homogeneous structure M is ω-categorical.
- Pouzet showed that set-homogeneous structures are uniformly prehomogeneous[1]
- Under the assumption of a finite relational language, a structure is uniformly prehomogeneous $\Leftrightarrow$ its theory is ω-categorical and model-complete.

[1] M is *uniformly prehomogeneous* if for every finite substructure F, there is finite G, of size bounded in terms of the size of F, such that $F \subseteq G \subseteq M$ and every embedding $F \to M$ which extends to an embedding $G \to M$ also extends to an automorphism of M.

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- Under the assumption of a finite relational language, a structure is uniformly prehomogeneous $\Leftrightarrow$ its theory is ω-categorical and model-complete.
- Thus, the permutation group-theoretic weakening

  homogeneous $\rightsquigarrow$ set-homogeneous

  relates to the model-theoretic weakening

  elimination of quantifiers $\rightsquigarrow$ model complete.

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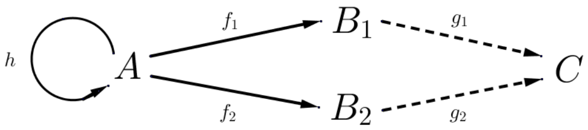
A Fraïssé-type theorem

Droste, Giraudet, Macpherson, Sauer (1994): Proved a set-homogeneous analogue of Fraïssé's theorem.

An age $\mathcal{A}$ has the twisted amalgamation property (TAP) if:

(TAP) Given structures $A, B_1, B_2 \in \mathcal{A}$ and embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathcal{A}$, $h \in \mathrm{Aut}(A)$, and embeddings $g_i : B_i \to C$ $(i = 1, 2)$ such that $f_1 \circ g_1 = h \circ f_2 \circ g_2$.



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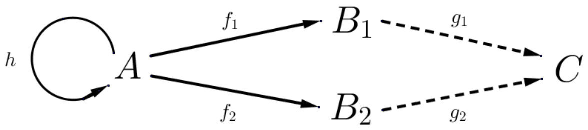

Theorem

(i) If M is a set-homogeneous structure then the age of M has TAP.

(ii) If $\mathcal{A}$ is an age which has TAP then there is a countable structure of age $\mathcal{A}$ which is set-homogeneous. Furthermore, any two such structures are isomorphic.

Classification results

Some classification results of homogeneous structures

	Finite	Countably infinite
Posets	(trivial)	Schmerl (1979)
Graphs	Gardiner (1976)	Lachlan & Woodrow (1980)
Tournaments	Woodrow (1976)	Lachlan (1984)
Digraphs	Lachlan (1982)	Cherlin (1998)

- What can we say about set-homogeneous posets, graphs, tournaments, digraphs, ...?
- Are classification results like those above still possible if we weaken homogeneity to set-homogeneity?

k-homogeneity and k-set-homogeneity

M - a relational structure

Definition

- M is k-homogeneous if every isomorphism between k-element substructures of M extends to an automorphism of M.
- M is $\leq k$-homogeneous if it is l-homogeneous for all $l \leq k$.

Definition

- M is k-set-homogeneous if, whenever U and V are isomorphic substructures of M of size k, there is an automorphism of M taking U to V.
- M is $\leq k$-set-homogeneous if it is l-set-homogeneous for all $l \leq k$.

Posets

Homogeneous posets

Schmerl (1979): Showed that the countable homogeneous posets are the (unique) universal one, and countably many others, built from $\mathbb{Q}$ and antichains in easily describable ways.

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Set-homogeneous posets

Droste (1985), (1987): Studied the structure of infinite posets (of arbitrary cardinality) which are k-set-homogeneous for some $k \geq 2$.

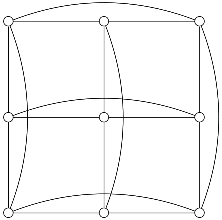
It follows from Droste's results any infinite set-homogeneous poset is homogeneous.

So, for posets homogeneity and set-homogeneity are equivalent.

Finite graphs

Homogeneous graphs

Gardiner (1976): Classified the finite homogeneous graphs, showing they are: C_5, $K_3 \times K_3$, and the rest are disjoint unions of complete graphs (or the complement).



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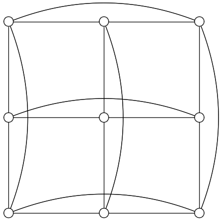

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Set-homogeneous graphs

Ronse (1978): Proved that for finite graphs homogeneity and set-homogeneity are equivalent.

- This was done by classifying the finite set-homogeneus graphs and then observing that they are all, in fact, homogeneous.

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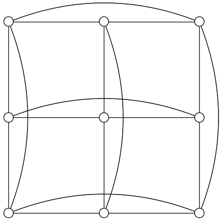

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Enomoto (1981): Gave a beautiful direct proof of the fact that for finite graphs set-homogeneous implies homogeneous.

So, for finite graphs homogeneity and set-homogeneity are equivalent.

Countably infinite graphs

Homogeneous graphs

Theorem (Lachlan and Woodrow (1980))

Let Γ be a countably infinite homogeneous graph. Then Γ is isomorphic to one of: the random graph, the generic K_n-free graph (or its complement), or a disjoint union of complete graphs (or its complement).

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Set-homogeneous graphs

Droste, Giraudet, Macpherson, Sauer (1994): Proved results about countably infinite set-homogeneous graphs.

In particular they constructed examples of countable set-homogeneous graphs that are not homogeneous.

Digraphs and tournaments

Definition (Digraph)

A digraph D is a relational structure $(D, \to)$ where $\to$ is an irreflexive antisymmetric binary relation.

So for any pair of vertices u, v exactly one of the following holds:

$$u \to v, \qquad v \to u, \qquad u \parallel v.$$

In particular, we do not allow loops and do not allow $u \leftrightarrow v$.

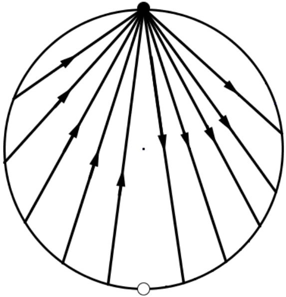
Definition (Tournament)

A tournament is a digraph where for any pair of vertices u, v either $u \to v$ or $v \to u$.

Circular digraphs

Homogeneous tournament $S(2)$

- **Vertex set**
  - countable dense subset of the unit circle
  - no two points make an angle of π at the centre of the circle
- **Directed edges**
  $x \to y$ iff $0 < \arg(x/y) < \pi$.



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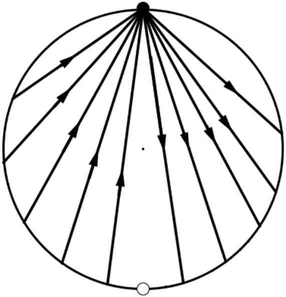

Homogeneous digraph $S(3)$

- **Vertex set**
  - countable dense subset of the unit circle
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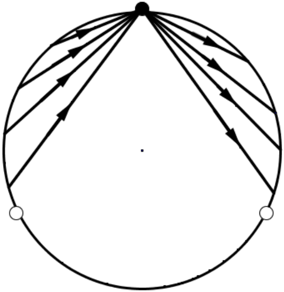
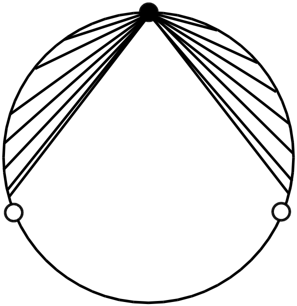


A circular graph

The circular structure $R(3)$

- **Vertex set**
  - countable dense subset of the unit circle
  - no two points make an angle of $2\pi/3$ at the centre of the circle
- **Edges**
  $x \sim y$ iff $0 < \arg(x/y) < 2\pi/3$.



- $R(3)$ is the graph obtained by taking the digraph $S(3)$ and replacing each directed edge by an undirected edge.

Properties of the graph $R(3)$

Theorem (Droste, Giraudet, Macpherson, Sauer (1994))

The graph $R(3)$ is set-homogeneous but not 3-homogeneous.

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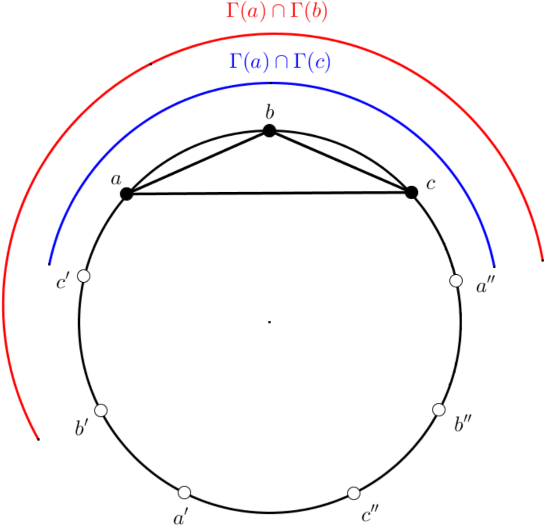

(I) Not 3-homogeneous

$\Gamma(a) \cap \Gamma(c) \subsetneq \Gamma(a) \cap \Gamma(b)$

It follows that there is no automorphism which:

Fixes a and swaps b and c.

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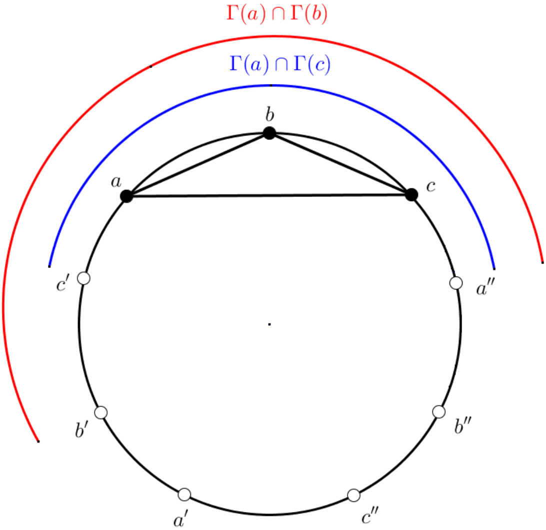

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(II) Set-homogeneous

Proof makes use of homogeneity of the closely related digraph $S(3)$.

Countable set-homogeneous graphs

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The graph $R(3)$ is set-homogeneous but not 3-homogeneous.

Moreover, any countably infinite ≤ 8-set-homogeneous graph which is not ≤ 3-homogeneous is isomorphic to $R(3)$ or its complement $\overline{R(3)}$.

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Problem

Are $R(3)$ and $\overline{R(3)}$ the only countable set-homogeneous graphs which are not homogeneous?

Finite tournaments

Homogeneous tournaments

Woodrow (1976): Classified the finite homogenoeus tournaments. There are just two of them: the one-vertex tournament, and the directed triangle D_3.

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Set-homogeneous tournaments

Lemma (RDG, Macpherson, Praeger, Royle (2012))

Let M be a finite set-homogeneous tournament. Then M is homogeneous, so is of size 1 or is isomorphic to D_3.

Proof: A straightforward adaptation of the argument Enomoto used to prove every finite set-homogeneous graph is homogeneous.

So, for finite tournaments homogeneity and set-homogeneity are equivalent.

Countably infinite tournaments

Homogeneous tournaments

Lachlan (1984): Completed the classification of countably infinite homogeneous tournaments (started by Woodrow). There are three countably infinite homogeneous tournaments: the generic tournament, the tournament $S(2)$, and the rationals $\mathbb{Q}$ (with $q_1 \to q_2$ iff $q_1 < q_2$).

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Proposition (RDG, Macpherson, Praeger, Royle (2012))

If T be a set-homogeneous tournament then T is ≤ 4-homogeneous.

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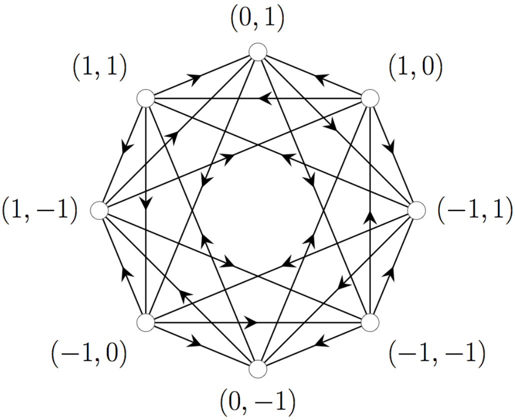
Problem

Is there a countable set-homogeneous tournament that is not homogeneous?

Finite homogeneous digraphs

Lachlan (1982): Classified the finite homogeneous digraphs, showing that they are:

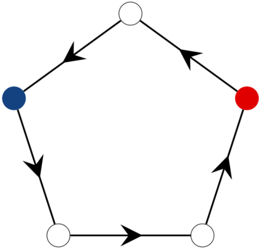
- directed cycles D_3, D_4,
- null digraphs $\overline{K}_n$,
- digraphs formed using the 'compositional product' $\overline{K}_n[D_3]$ and $D_3[\overline{K}_n]$ $(1 \leq n)$,
- the sporadic example H_0.



(**Note:** In fact Lachan proved quite a lot more than this, classifying finite homogeneous digraphs where one also allows $u \leftrightarrow v$.)

A finite set-homogeneous non-homogenoeus digraph

Example



The directed 5-cycle:

- is a set-homogeneous digraph
- is not homogeneous: (•, •) $\mapsto$ (•, •) does not extend to an automorphism

Finite set-homogeneous digraphs

Theorem (RDG, Macpherson, Praeger, Royle (2012))

Let D be a finite set-homogeneous digraph. Then either D is homogeneous or it is isomorphic to D_5.

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- Carry out the classification of finite set-homogeneous digraphs.
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Problem

Give a short direct proof of this theorem.

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Set-homogeneous digraphs

RDG, Macpherson, Praeger, Royle (2012): We have established some initial results on countable set-homogeneous digraphs.

As in the work of Droste, Giraudet, Macpherson, Sauer (1994), circular structures can be used to obtain examples.

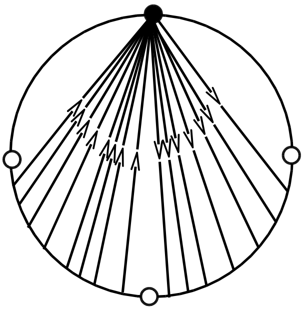
In contrast to the situation for graphs, we have countably infinitely many distinct examples of countable set-homogeneous digraphs that are not homogeneous.

$T(4)$: a countable set-homogeneous digraph

Definition

Let $T(4)$ be the digraph with

- **Vertex set**
  - countable dense subset of the unit circle
  - no two points make an angle of of π or $\pi/2$ at the centre
- **Directed edges**
  $x \to y$ iff $\pi/2 < \arg(x/y) < \pi$.



Properties of $T(4)$

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The digraph $T(4)$ is set-homogeneous but not 2-homogeneous.

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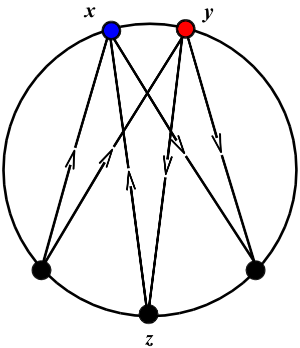
- **(I) Set-homogeneous:** Proof makes use of fact that $T(4)$ can be expanded to a homogeneous structure (the binary structure $S(4)$ introduced by Cameron).
- **(II) Not 2-homogeneous:** There exist independent pairs that cannot be swapped by any automorphism:

e.g. if $x, y \in T(4)$ with $0 < \arg(x/y) < \pi/2$, then

- $\exists z(z \to x \wedge y \to z)$ but
- $\neg\exists z(z \to y \wedge x \to z)$.

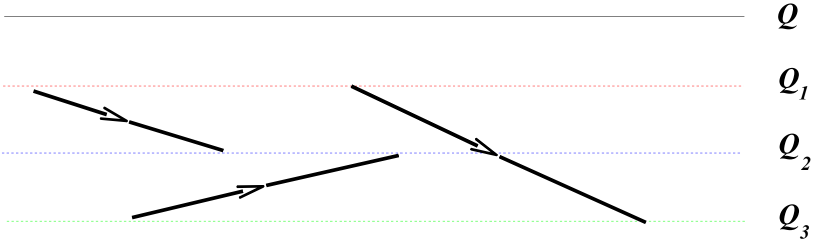


R_n $(n \geq 2)$: a family of set-homogeneous digraphs

Definition

- Let $2 \leq n \leq \aleph_0$
- let $\{Q_i : i < n\}$ be a partition of $\mathbb{Q}$ into n dense codense sets.

Define a digraph R_n with domain $\mathbb{Q}$, putting $a \to b$ if and only if $a < b$ and there is no $i < n$ such that $a, b \in Q_i$.



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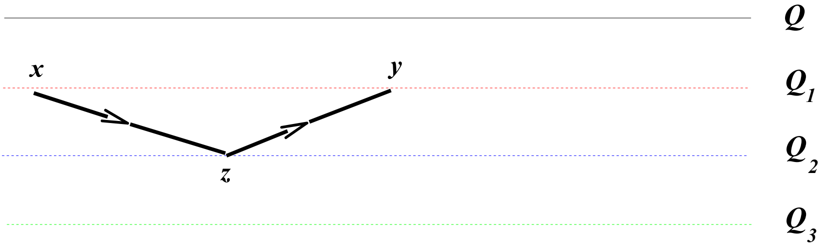
  e.g. if $x, y \in Q_1$ with $x < y$ then there is z with $x \to z \to y$ but no z with $y \to z \to x$. Thus $(x, y) \mapsto (y, x)$ does not extend to an automorphism.



Countable set-homogeneous digraphs

Theorem (RDG, Macpherson, Praeger, Royle (2012))

Let D be a countably infinite set-homogeneous digraph which is not 2-homogeneous. Then D is isomorphic to $T(4)$ or to R_n for some $n \geq 2$.

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Problem

Are $T(4)$ and R_n $(n \geq 2)$ the only countably infinite set-homogeneous digraphs which are not homogeneous?

Notes on the proof...

Definition: G - a transitive permutation group acting on a set Ω.
An orbital of G is a G-orbit Λ on $\Omega \times \Omega$.
$\Lambda(\alpha) := \{x : (\alpha, x) \in \Lambda\}$ the 'out-neighbours' of α.
$\Lambda^* := \{(y, x) : (x, y) \in \Lambda\}$ the orbital paired to Λ.

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$\Lambda^* := \{(y, x) : (x, y) \in \Lambda\}$ the orbital paired to Λ.

D - a set-homogeneous digraph a countably infinite set-homogeneous digraph which is not 2-homogeneous.

There are orbitals:

- Γ such that $(\alpha, \beta) \in \Gamma$ iff $\alpha \to \beta$
- Λ is an orbital such that $(\alpha, \beta) \in \Lambda$ implies $\alpha \neq \beta$ and $\alpha \parallel \beta$.

Saying D not 2-homogeneous is equivalent to saying Λ is not self-paired.

Notes on the proof...

Definition: G - a transitive permutation group acting on a set Ω.
An orbital of G is a G-orbit Λ on $\Omega \times \Omega$.
$\Lambda(\alpha) := \{x : (\alpha, x) \in \Lambda\}$ the 'out-neighbours' of α.
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Notation: Write $\alpha \Rightarrow \beta$ iff $(\alpha, \beta) \in \Lambda$.

For all vertices $\alpha \neq \beta$ exactly one of the following holds:

$$\begin{array}{llll} \alpha \to \beta & (\alpha, \beta) \in \Gamma, & \alpha \leftarrow \beta & (\alpha, \beta) \in \Gamma^*, \\ \alpha \Rightarrow \beta & (\alpha, \beta) \in \Lambda, & \alpha \Leftarrow \beta & (\alpha, \beta) \in \Lambda^*. \end{array}$$

The primitive case

D - a set-homogeneous digraph a countably infinite set-homogeneous digraph which is not 2-homogeneous. Suppose $G = \mathrm{Aut}(D)$ is primitive.

We prove:

- $\Gamma(\alpha)$, $\Gamma^*(\alpha)$, $\Lambda(\alpha)$ and $\Lambda^*(\alpha)$ are all infinite,

and whenever $\alpha \Rightarrow \beta$ we have

- $\Gamma(\alpha) \neq \Gamma(\beta)$ and $\Gamma^*(\alpha) \neq \Gamma^*(\beta)$, and
- $\Gamma(\alpha) \setminus \Gamma(\beta)$, $\Gamma(\beta) \setminus \Gamma(\alpha)$, $\Gamma^*(\alpha) \setminus \Gamma^*(\beta)$, and $\Gamma^*(\beta) \setminus \Gamma^*(\alpha)$ are all non-empty.
  - (Uses Droste's classification of 2-set-homogeneous semilinear orders.)

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From this it follows that if $\beta \Rightarrow \alpha$ then either

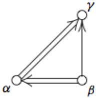
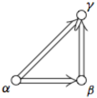
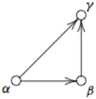
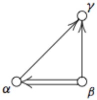
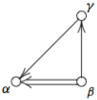
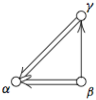
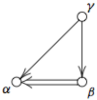
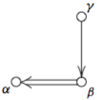
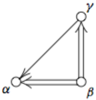
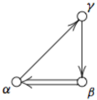
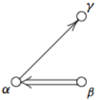
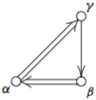
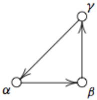
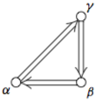
(a) there exists γ with $\beta \to \gamma$ and $\alpha \parallel \gamma$, or

(b) there exists δ with $\alpha \to \delta$ and $\beta \parallel \delta$.

WLOG we may suppose that (a) holds (this can be used to then show (b) does not hold, compare L_1 and L_9 below).

Lemma: L_1–L_6 embed while L_7–L_{14} do not.

Table 1
Configurations on 3 vertices.

L_1		L_6		L_{11}	
L_2		L_7		L_{12}	
L_3		L_8		L_{13}	
L_4		L_9		L_{14}	
L_5		L_{10}			

Recovering the circular structure $T(4)$

The above configurations embedding lemma then implies:

- $\Gamma(\alpha)$, $\Gamma^*(\alpha)$, $\Lambda(\alpha)$ and $\Lambda^*(\alpha)$ are linearly ordered by $\Rightarrow$ densely and without endpoints.

Fix $\alpha \in D$. There is a unique linear ordering $<_\alpha$ on D with

- convex subsets $\Lambda(\alpha)$, $\Lambda^*(\alpha)$, $\Gamma(\alpha)$, $\Gamma^*(\alpha)$ where $<_\alpha$ extends $\Rightarrow$, and
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There is a corresponding circular ordering K_α on D

$$K_\alpha(x,y,z) \text{ iff } x <_\alpha y <_\alpha z \text{ or } y <_\alpha z <_\alpha x \text{ or } z <_\alpha x <_\alpha y.$$

It may then be shown that:

- K_α is independent of the choice of α, so may be denoted by K.
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- K is $\mathrm{Aut}(D)$-invariant.

Let D, D' be countable set-homogeneous not 2-homogeneous digraphs with primitive automorphism groups. We then prove $D \cong D'$ using the above structural information to give a back and forth argument.

The imprimitive case

Lemma

Let D be a countably infinite connected set-homogeneous digraph and suppose that $G = \mathrm{Aut}(D)$ preserves a nontrivial block system $\{B_i : i \in I\}$. Then one of the following holds, for some set-homogeneous tournament T and $n \leq \aleph_0$.

(i) $M \cong \overline{K_n}[T]$, D is 2-homogeneous, and each B_i induces T;

(ii) $M \cong T[\overline{K_n}]$, D is 2-homogeneous, and each $B_i \cong \overline{K_n}$;

(iii) Each B_i is an independent set, and for distinct $x, y \in B_i$, $\Gamma(x) \neq \Gamma(y)$; also, for distinct $i, j \in I$, there are arcs in both directions between B_i and B_j.

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D - countably infinite set-homogeneous digraph which is not 2-homogeneous. Suppose $G = \mathrm{Aut}(D)$ is imprimitive.

Since D is not 2-homogeneous, case (iii) of the above Lemma must hold.

The imprimitive case

D - a countably infinite set-homogeneous digraph which is not 2-homogeneous. Suppose $G = \mathrm{Aut}(D)$ is imprimitive.
$\{B_i : i \in I\}$ - a non-trivial block system for G.

The G-action induced on B_i is highly homogeneous but not 2-transitive. An unpublished result of J. P. J. MacDermott then implies:

- G preserves a dense linear order without endpoints $<_i$ on B_i.
- Let $x, y \in B_i$ with $x <_i y$. Since $\Gamma(x) \neq \Gamma(y)$, either there is z with $x \to z \to y$ or there is w with $y \to w \to x$ (but only one of z or w can exist).

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- Thus, reversing some of the $<_i$ if necessary, we may suppose that $x <_i y$ if and only if $\Gamma(x) \supset \Gamma(y)$.
- Define $<$ on D by $x < y$ iff $\quad x \to y \quad$ or $\quad x \parallel y$ and $\Gamma(x) \supset \Gamma(y)$.

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- Define $<$ on D by $x < y$ iff $\;x \to y\;$ or $\;x \parallel y$ and $\Gamma(x) \supset \Gamma(y)$.
- Then we show $(D, <)$ is dense without endpoints and with the B_i dense codense, and thus recover the structure R_n.

Further problems

Binary relations

An s-digraph is the same as a digraph except that we allow pairs of vertices to have arcs in both directions i.e. we allow $u \leftrightarrow v$.

- Lachlan (1982): Classified the finite homogeneous s-digraphs.
- RDG, Macpherson, Praeger, Royle (2012): Classified the finite set-homogeneous s-digraphs. We obtain Lachlan's result as a corollary.

Problem

Compare homogeneity and set-homogeneity for countably infinite s-digraphs.

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Homegenizability
A structure is homogenizable if it can be made homogeneous by adding finitely many relations to the language (without changing the automorphism group). Droste, Giraudet, Macpherson, Sauer (1994) pose the problem:

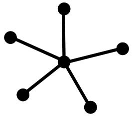
Problem

Is there a set-homogeneous structure which is not homogenizable?

Some graph theoretic terminology and notation

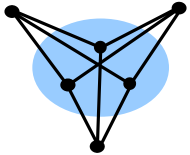
Definition
$\Gamma = (V\Gamma, \sim)$ - a graph

So $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$



- Let v be a vertex of Γ. Then the neighbourhood $\Gamma(v)$ of v is the set of all vertices adjacent to v. So

$$\Gamma(v) = \{w \in V\Gamma : w \sim v\}$$



- For $X \subseteq V\Gamma$ we define

$$\Gamma(X) = \{w \in V\Gamma : w \sim x \; \forall x \in X\}$$

Enomoto's argument

Lemma (Enomoto's lemma)

Let Γ be a finite set-homogeneous graph and let U and V be induced subgraphs of Γ. If $U \cong V$ then $|\Gamma(U)| = |\Gamma(V)|$.

Proof.

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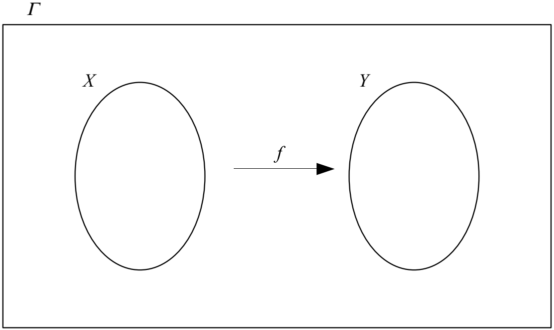
- Then $(\Gamma(U))g = \Gamma(V)$.
- In particular $|\Gamma(U)| = |\Gamma(V)|$.

Enomoto's argument

Γ - finite set-homogeneous graph X, Y - induced subgraphs
$f : X \to Y$ an isomorphism



Claim: The isomorphism $f : X \to Y$ is either an automorphism, or extends to an isomorphism $f' : X' \to Y'$ where $X' \supsetneq X$ and $Y' \supsetneq Y$.

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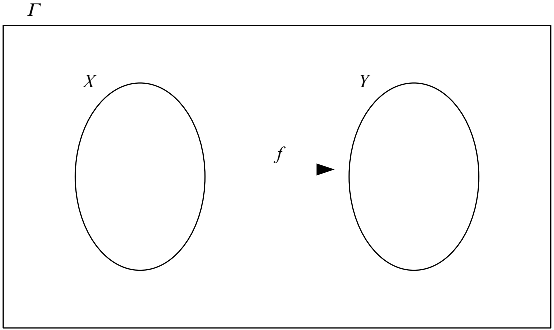

Proof of claim.

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.

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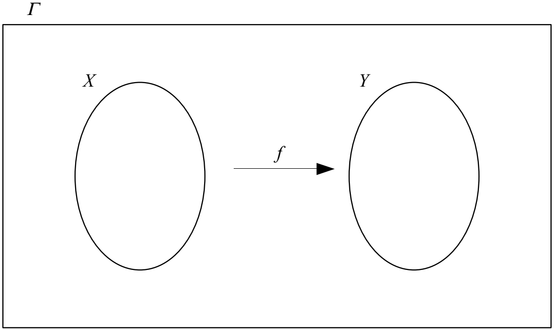

Proof of claim.

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.
- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.

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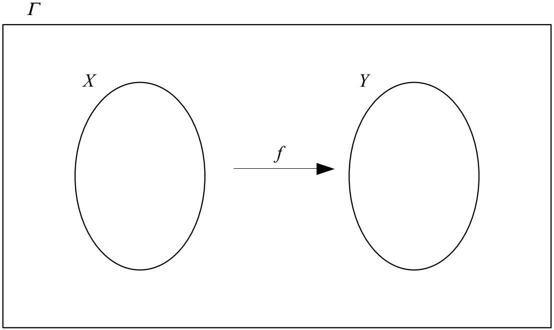

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- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.
- Suppose $|\Gamma(a) \cap X| \geq |\Gamma(d) \cap Y|$ (the other possibility is dealt with dually using the isomorphism f^{-1})

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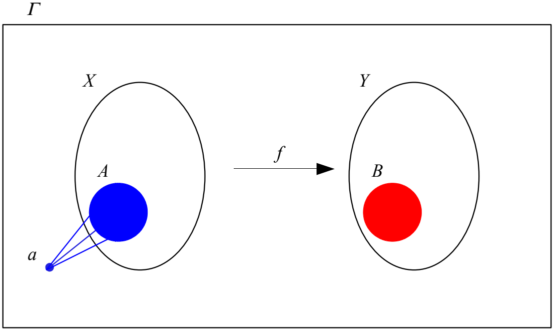

Proof of claim.

- Let $A = \Gamma(a) \cap X$ and define $B = f(A)$.

Enomoto's argument

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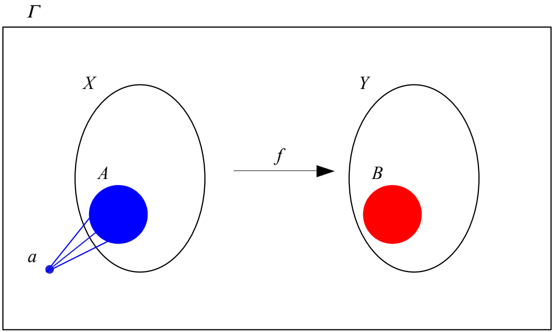

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- $A \cong B$ & Γ is set-homogeneous so by the lemma $|\Gamma(A)| = |\Gamma(B)|$.

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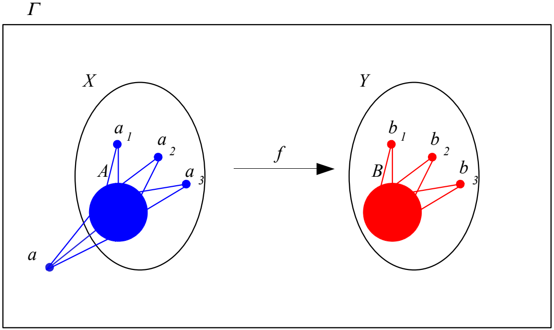

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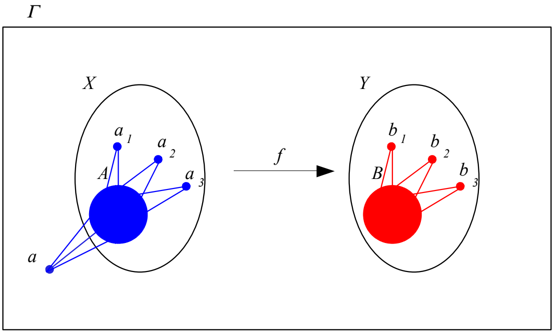

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- $\Gamma(B) \cap Y = f(\Gamma(A) \cap X)$ so $|\Gamma(B) \cap Y| = |\Gamma(A) \cap X|$.
- $\therefore |\Gamma(B) \setminus Y| = |\Gamma(A) \setminus X)| \geq 1$

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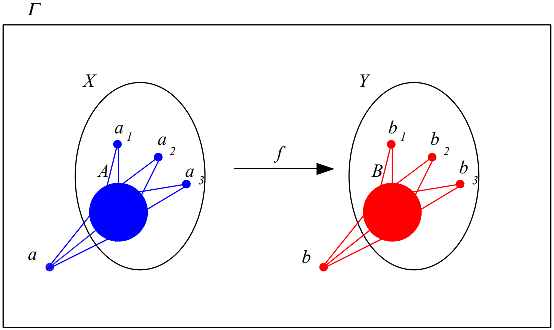

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- Let $b \in \Gamma(B) \setminus Y$ and extend f to $f' : X \cup \{a\} \to Y \cup \{b\}$ by defining $f'(a) = b$.

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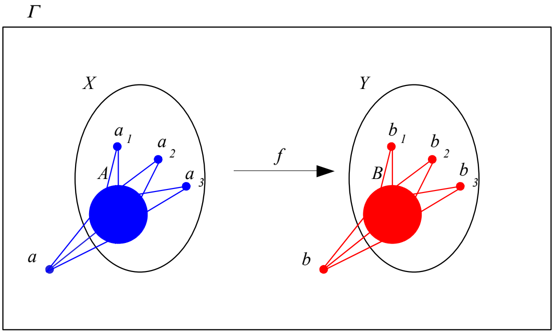

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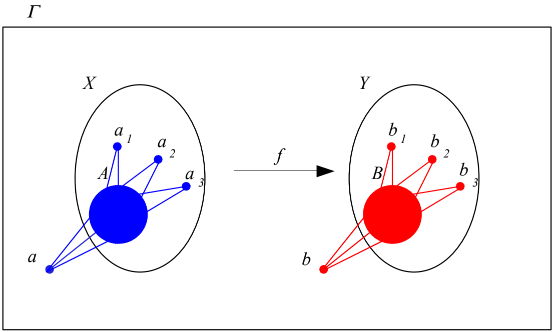

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- $\Gamma(b) \cap Y = B$ by maximality in original definition of a,
- $\therefore f'$ is an isomorphism.

Set-homogeneous digraphs

Question: Does Enomoto's argument apply to other kinds of structure?

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Definition (Digraphs)

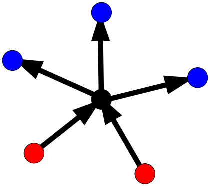
A digraph D consists of a set VD of vertices together with an irreflexive antisymmetric binary relation $\to$ on VD.

Definition (in- and out-neighbours)

A vertex $v \in VD$ has a set of in-neighbours and a set of out-neighbours

$$D^+(v) = \{w \in VD : v \to w\}, \quad D^-(v) = \{w \in VD : w \to v\}.$$

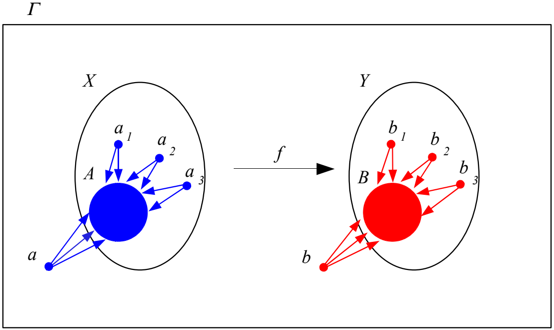
A vertex with red in-neighbours and blue out-neighbours



Enomoto's argument for digraphs

D - finite set-homogeneous digraph $\quad X, Y$ - induced subdigraphs

$f : X \to Y$ an isomorphism



- Follow the same steps but using out-neighbours instead of neighbours.
- Everything works, except the very last step.

Enomoto's argument for digraphs

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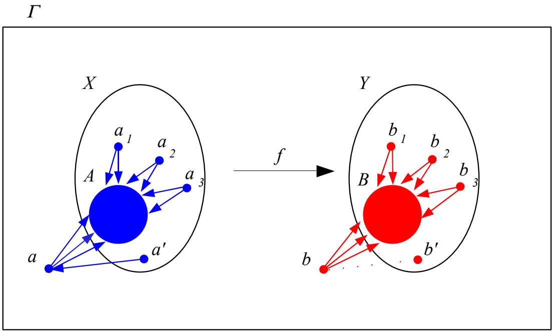

- Follow the same steps but using out-neighbours instead of neighbours.
- Everything works, except the very last step.
- We do not know how b is related to the vertices in the set $Y \setminus B$.
  So f' may not be an isomorphism.

Enomoto's argument for digraphs

The key point:

- For graphs, given $u, v \in V\Gamma$ there are 2 possibilities

$$u \sim v \quad \text{or} \quad u \parallel v \ \ \text{(meaning that } u \text{ \& } v \text{ are unrelated)}.$$

- For digraphs, given $u, v \in VD$ there are 3 possibilities

$$u \to v \quad \text{or} \quad v \to u \quad \text{or} \quad u \parallel v.$$

Enomoto's argument for digraphs

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However, the argument does work for tournaments:

Definition

A tournament is a digraph where for any pair of vertices u, v either $u \to v$ or $v \to u$.

Corollary

Let T be a finite tournament. Then T is homogeneous if and only if T is set-homogeneous.