Constraint Satisfaction Problems for Reducts of Homogeneous Structures

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   - Automatic continuity
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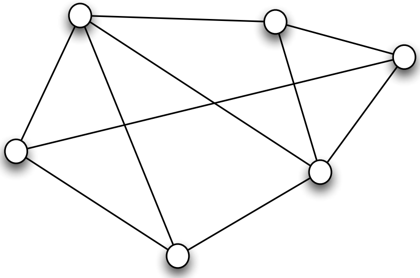

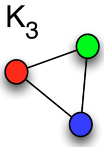

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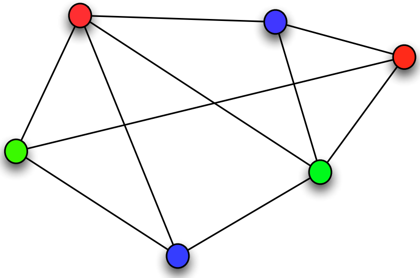

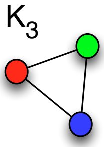

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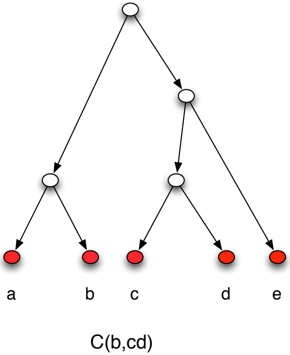

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C2. $\forall a, b, c \left(C(a, bc) \Rightarrow \neg C(b, ac) \right)$

C3. $\forall a, b, c, d \left(C(a, bc) \Rightarrow C(a, dc) \vee C(d, bc) \right)$

C4. $\forall a, b \left(a \neq b \Rightarrow C(a, bb) \right)$

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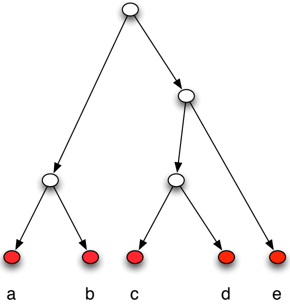

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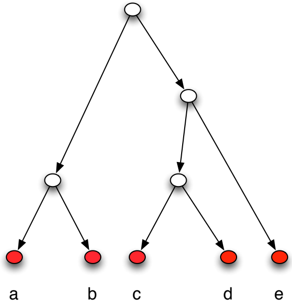

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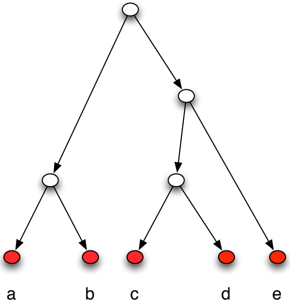

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CSP(Γ): rooted triple consistency problem is in P (Aho+Sagiv+Szymanski+Ullman'81)

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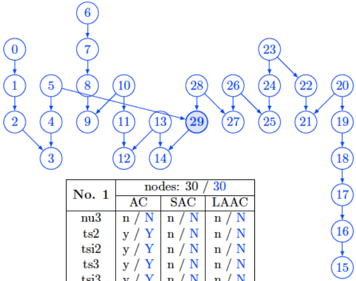

No. 1	nodes: 30 / 30		
	AC	SAC	LAAC
nu3	n / N	n / N	n / N
ts2	y / Y	n / N	n / N
tsi2	y / Y	n / N	n / N
ts3	y / Y	n / N	n / N
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sig4	Y	N	-

⇒ NP-complete.

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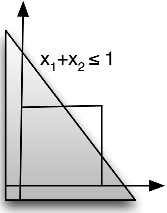

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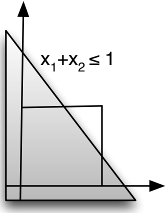

The set of all polymorphisms Pol(Γ) of Γ is a function clone $\mathcal{C}$:

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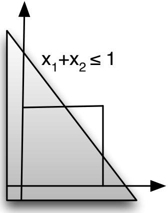

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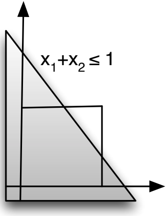

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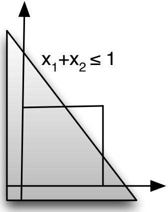

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A topological clone isomorphism must additionally be a homeomorphism.

Topological Clones and Interpretations

Theorem (B+Pinsker'13).

Let Γ and Δ be ω-categorical. Then Pol(Γ) and Pol(Δ) are topologically isomorphic if and only if Γ and Δ are primitive positive bi-interpretable.

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- Under which conditions are homomorphisms from Pol(Γ) to Pol(Δ), for finite Δ, continuous?

"Automatic continuity"

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In each case, CSP(Γ) is in P iff Γ has a polymorphism f of arity $n \geq 2$ and endomorphisms $e_1, \dots, e_n$ satisfying

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This polymorphism condition is not sufficient for tractability of CSP(Γ) but it might be under additional assumptions on Γ (that can be made wlog) $\rightarrow$ talk of Michael Pinsker.

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Reference: Pinsker+B'12, Reducts of Ramsey structures.

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In the case of reducts Γ of $(\mathbb{Q}; <)$: other ideas are needed.

Grohe's Datalog Hierarchy Theorem via Homogeneous Structures

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- For finite binary structures Γ, we have (Barto+Kozik'09)

$$\text{CSP}(\Gamma) \in \text{Datalog}_k \quad \Rightarrow \quad \text{CSP}(\Gamma) \in \text{Datalog}_2$$

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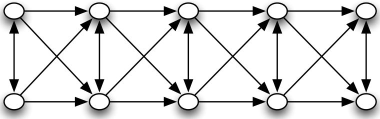

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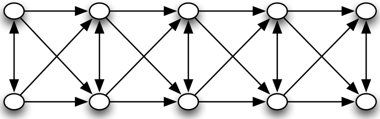

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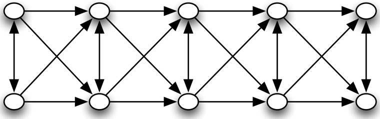

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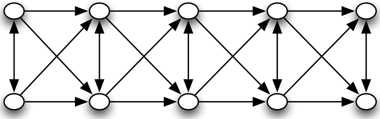

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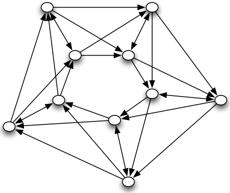
k-cycle: k-walk $(x^0, x^1, \ldots, x^m)$ where $x^m = x^0$.

Homogenisation

$\mathcal{C}$: class of digraphs without k-cycle.



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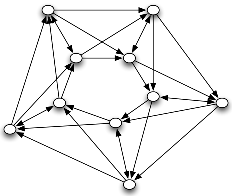

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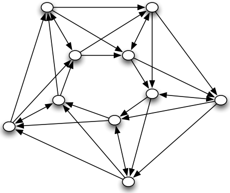

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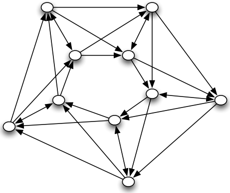

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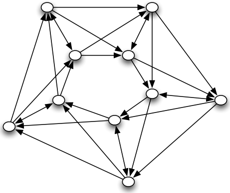

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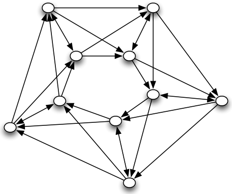

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- Γ is an ω-categorical digraph with the desired properties:
  - CSP(Γ) is in Datalog$_{2k}$
  - CSP(Γ) is not in Datalog$_{2k-1}$

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Lemma

$\mathcal{D}$ is an amalgamation class.

Lower Bound

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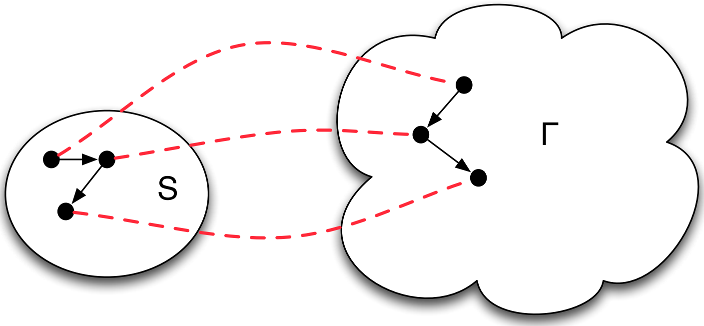
Theorem (Dalmau-B.'05).

Let Γ be ω-categorical with finite relational signature τ. Then CSP(Γ) is not in Datalog$_{2k-1}$ if and only if for for every $l \in \mathbb{N}$ there exists a finite τ-structure S such that Duplicator wins the existential $(2k-1, l)$-pebble game on (S, Γ).



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4. What about the Ramsey properties of my ω-categorical digraphs from the Datalog hierarchy?