Littlewood polynomials with L_4 norms *invariant* under rotations of the coefficients

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BIRS Workshop
The Geometry, Algebra and Analysis of Algebraic Numbers

Norms of Littlewood Polynomials

Let the set of all **unimodular polynomials** of degree $n-1$ be denoted by

$$\mathfrak{U}_n := \left\{P(z) : P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}, \right.$$
$$\left. a_\ell \in \mathbb{C}, |a_\ell| = 1, \forall 0 \le \ell \le n-1\right\}$$

and also the set of all **Littlewood polynomials** of degree $n-1$ be denote by

$$\mathscr{L}_n := \left\{P(z) : P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}, \right.$$
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Recall that the L_p norm of $P(z)$ is given by

$$\|P\|_p := \left(\int_0^1 \left|P(e^{2\pi i\theta})\right|^p d\theta\right)^{1/p}.$$

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Kahane's Theorem and Ultra-Flat Polynomials

In this talk, we are particularly interested in L_4 norms.

Also note that $||P||_2 = \sqrt{n}$ for all $P(z) \in \mathfrak{U}_n$.

Kahane's Theorem (1980)

There is a sequence of $\{P_n(z)\}_{n=1}^{\infty}$ in $\mathfrak{U}_n$ satisfying

$$\left|P_n(z)\right| = (1 + o(1))\sqrt{n}, \text{ for all } |z| = 1.$$

We call such $P_n(z)$ **Ultra-Flat Polynomial.**

Open Question: *Do Littlewood ultra-flat polynomials exist or not? Is there a sequences of* $\{P_n(z)\}_{n=1}^{\infty}$ *in* $\mathscr{L}_n$ *satisfying*

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Erdös and L_4 Norm Conjectures

Erdös Conjecture (1957)

There is $\varepsilon > 0$, such that

$$(1+\varepsilon)\sqrt{n} \leq \left|P(z)\right|, \forall |z| = 1$$

for any $P(z) \in \mathscr{L}_n$.

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There is $\varepsilon > 0$, such that

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Erdös Conjecture $\Rightarrow$ L_4 Norm Conjecture $\Rightarrow$ No Ultra-Flat Littlewood Polynomial exists!

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Finding Littlewood Polynomials with Small L_4 norm

Both Erdös and L_4 norm conjectures are still wide open and we are interested in finding sequences of Littlewood polynomials having small L_4 norm.

Let p be an odd prime. **Fekete** Polynomial $F_p(z)$ is the Littlewood polynomial of degree $p-1$ defined by

$$F_p(z) = 1 + \left(\frac{1}{p}\right) z + \left(\frac{2}{p}\right) z^2 + \cdots + \left(\frac{p-1}{p}\right) z^{p-1}$$

where $\left(\frac{\cdot}{p}\right)$ is the usual Legendre symbols.

It was proved by Hoholdt and Jensen that

$$\|F_p(z)\|_4^4 = \left(\frac{5}{3} + o(1)\right) p^2.$$

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Rotation of the Coefficients of Littlewood Polynomials

For any $r \in (0,1]$, we define the polynomials constructed by rotating the coefficients to the right by $\lfloor rn \rfloor$ and denote by

$$(P)_r := \sum_{\ell=0}^{n-1} a_{\ell+\lfloor rn \rfloor \pmod n} z^n.$$

We simply call this polynomial a ***r*-rotation** of $P(z)$. Usually, the rotation will change the L_4 norm of $P(z)$, even the main term.

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Rotation of the Coefficients of Littlewood Polynomials

Hoholdt and Jensen (1988)

The $1/4$-rotation of Fekete polynomials,

$$R(z) = (F_p)_{1/4}(z) = \sum_{\ell=0}^{n-1} \left(\frac{\ell + \lfloor n/4 \rfloor}{p} \right) z^n$$

has

$$\|R\|_4^4 = \left(\frac{7}{6} + o(1) \right) p^2 \quad \left(< \left(\frac{5}{3} + o(1) \right) p^2 \right).$$

For more than 20 years, the value $7/6$ has remained the smallest known asymptotic L_4 norm of Littlewood polynomials.

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Truncation of Littlewood Polynomials

In 2003, Borwein, Jedwab and C. studied how the truncations and rotations change of the values of the L_4 norm of Littlewood polynomials.

For any $t \in (0,1]$, we define the ***t*-truncation** of polynomial, P, by

$$(P)^t := \sum_{\ell=0}^{\lfloor tn \rfloor} a_\ell z^n.$$

We compute the L_4 norms for various t-truncations and r-rotations of the Fekete polynomials for some large prime p, i.e., compute

$$\left\| \left((F_p)_r \right)^t \right\|_4^4, \qquad \text{for } 0 < r, t \leq 1.$$

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Merit Factors

In order to make computation and analysis of the data easier, instead of computing the L_4 norm, we compute its normalized version, so-called the merit factor:

Definition of Merit Factor.

The **merit factor** of polynomials $P(z)$ in $\mathcal{L}_n$ is defined by

$$MF(P) := \frac{n^2}{\|P(z)\|_4^4 - n^2} = \frac{\|P\|_2^4}{\|P\|_4^4 - \|P\|_2^4}.$$

Note that for a sequence $\{P_n\}$ in $\mathcal{L}_n$, if $\lim_{n\to\infty} MF(P_n) = c$, then

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Binary sequences and Aperiodic Autocorrections

Let

$$X = (a_0, a_1, \ldots a_{n-1})$$

be a finite binary ± 1 sequence of length n.

Aperiodic Autocorrections

The **(aperiodic) autocorrections**, $C_X(u)$, of finite binary sequence X are given by

$$C_X(u) := \sum_{\ell=0}^{n-u-1} a_\ell a_{\ell+u}, \quad , \text{ for } u = 0, 1, \ldots, n-1 \; .$$

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Merit Factors has many applications and are widely studied in the communication and Information theory.

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Rotations and Truncations of Legendre Sequences

We computed the r-rotation and t-truncation of Fekete polynomials of prime p. For any $0 < r, t \leq 1$, we let

$$f(r,t) := \lim_{p\to\infty} \left(\frac{1}{MF\left((F_p)_r^t\right)} \right)$$

and the r-rotation and t-truncation of Fekete polynomials of prime p is

$$(F_p)_r^t(z) = \sum_{\ell=0}^{\lfloor tp \rfloor} \left(\frac{\ell + \lfloor rp \rfloor}{p} \right) z^\ell.$$

L_4 Norms of Rotations and Truncations of Fekete Polynomials

Based on the numerical data, we made the following conjecture:

Borwein, Jedwab and C.'s Conjecture

For $0 < t \le \frac{1}{2}$,

$$f(r,t) = \begin{cases} 1 - \frac{4}{3}t & \text{for } 0 \le r \le \frac{1}{2} - t; \\ 1 - \frac{4}{3}t + \frac{4(r-\frac{1}{2}+t)^2}{t^2} & \text{for } \frac{1}{2} - t \le r \le \frac{1-t}{2}; \\ 1 - \frac{4}{3}t + \frac{4(r-\frac{1}{2})^2}{t^2} & \text{for } \frac{1-t}{2} \le r \le \frac{1}{2}; \end{cases}$$

and for $\frac{1}{2} < t \le 1$,

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Borwein, Jedwab and C.'s Conjecture

- However these Littlewood polynomials $(F_p)_r^t$ still don't give us smaller asymptotic L_4 norm! We need to append $(F_p)_r^t$ to the end of $(F_p)_r$, namely,

$$(F_p(z))_r + z^p(F_p(z))_r^t$$

and its merit factor can be easily determined by $f(r,t)$.

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Suppose the above conjecture is true. Then the maximum of $\lim_{p\to\infty} MF((F_p)_r + z^p(F_p)_r^t)$ over $r, t \in [0,1]$ is given by

$$\begin{aligned} \hat{r} &\simeq 0.2211 \\ \hat{t} &\simeq 0.0578 \\ \lim_{p\to\infty} MF\left((F_p)_{\hat{r}} + z^p(F_p)_{\hat{r}}^{\hat{t}}\right) &\simeq 6.3421. \end{aligned}$$

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Here $\hat{t} \simeq 0.0578$ is the root in $[0,1]$ of the cubic equation

$$4t^3 + 12t^2 - 18t + 1 = 0$$

and $\hat{r} = \frac{1}{4} - \frac{\hat{t}}{2} \simeq 0.2211$.

Jedwab, Katz and Schmidt's Result

Jedwab, Katz and Schmidt (2014)

Borwein, Jedwab and C.'s conjecture is true, that is, for $0 < t \le \frac{1}{2}$,

$$f(r,t) = \begin{cases} 1 - \frac{4}{3}t & \text{for } 0 \le r \le \frac{1}{2} - t; \\ 1 - \frac{4}{3}t + \frac{4(r-\frac{1}{2}+t)^2}{t^2} & \text{for } \frac{1}{2} - t \le r \le \frac{1-t}{2}; \\ 1 - \frac{4}{3}t + \frac{4(r-\frac{1}{2})^2}{t^2} & \text{for } \frac{1-t}{2} \le r \le \frac{1}{2}; \end{cases}$$

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New Record of Largest Merit Factor

The above Corollary gives

Jedwab, Katz and Schmidt (2014)

The smallest known asymptotic L_4 for a sequence of Littlewood polynomial is:

$$\begin{aligned} \left\| (F_p(z))_{\hat{r}} + z^p (F_p)_{\hat{r}}^{\hat{t}} \right\|_4^4 &= \left(1 + \frac{1}{6.3421} + o(1)\right) p^2 \\ &= (1.15767\cdots + o(1))\, p^2 \\ &< \left(\frac{7}{6} + o(1)\right) p^2 = (1.16666\cdots + o(1))\, p^2. \end{aligned}$$

Jedwab, Katz and Schmidt's Result

Jedwab, Katz and Schmidt (2014)

For $0 < t \le \frac{1}{2}$,

$$f(r,t) = \begin{cases} 1 - \frac{4}{3}t & \text{for } 0 \le r \le \frac{1}{2} - t; \\ 1 - \frac{4}{3}t + \frac{4(r-\frac{1}{2}+t)^2}{t^2} & \text{for } \frac{1}{2} - t \le r \le \frac{1-t}{2}; \\ 1 - \frac{4}{3}t + \frac{4(r-\frac{1}{2})^2}{t^2} & \text{for } \frac{1-t}{2} \le r \le \frac{1}{2}; \end{cases}$$

and for $\frac{1}{2} < t \le 1$,

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$$f(r,t) = 1 - \frac{4}{3}t, \qquad \text{is independent of the rotation } r.$$

Corollary

For $0 < t \leq \frac{1}{2}$ and $0 \leq r \leq \frac{1}{2} - t$, we have

$$\begin{aligned} \|(F_p)_r^t\|_4^4 &= \left\| \sum_{\ell=0}^{\lfloor tp \rfloor} \left(\frac{\ell + \lfloor rp \rfloor}{p} \right) z^\ell \right\|_4^4 \\ &= (1+o(1)) \left(2 - \frac{4}{3}t\right)(tp)^2, \qquad \text{is independent of } r. \end{aligned}$$

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Jedwab, Katz and Schmidt's Result

The coefficients of $(F_p)_r^t$ are

$$\left(\frac{\lfloor rp \rfloor}{p}\right), \left(\frac{1+\lfloor rp \rfloor}{p}\right), \cdots, \left(\frac{(\lfloor tp \rfloor - 1) + \lfloor rp \rfloor}{p}\right).$$

For $t = 1/4$ and $0 \leq r \leq 1/4$,

Jedwab, Katz and Schmidt's Result

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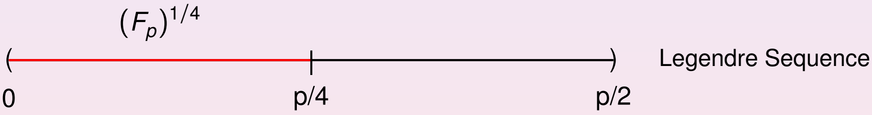
For $t = 1/4$ and $r = 0$,



$$\|(F_p)^{1/4}\|_4^4 = \left(\frac{5}{3} + o(1)\right)\left(\frac{p}{4}\right)^2.$$

Jedwab, Katz and Schmidt's Result

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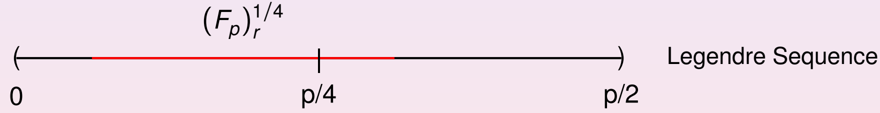

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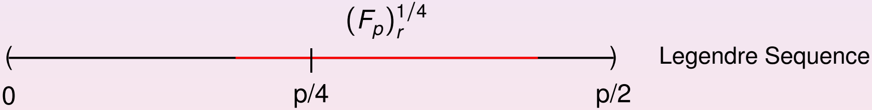

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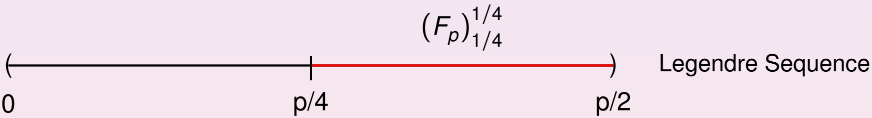

$$\|(F_p)_{1/4}^{1/4}\|_4^4 = \left(\frac{5}{3} + o(1)\right)\left(\frac{p}{4}\right)^2.$$

Average Value of Merit Factors Binary Sequences

Borwein and Lockhart (2000)

The expected value of $\frac{1}{MF(X)}$ for all binary sequences X of length n is

$$E\left(\frac{1}{MF(X)}\right) = 1.$$

and

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For every $\epsilon > 0$ and

$$\lim_{n\to\infty} \frac{1}{2^n}\left|\left\{X \in \mathscr{L}_n : \left|\frac{1}{MF(X)} - 1\right| > \epsilon\right\}\right| = 0$$

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Distribution of Legendre Sequences

• Does the corollary give any information about the distribution of Legendre sequences?

Conrey, Granville, Poonen and Soundararajan (Weil)

Fix integer J, and then the numbers $\delta_j \in \{-1, 1\}$ for each j with $|j| < J$. We have, uniformly,

$$\#\left\{x \pmod p : \left(\frac{x-j}{p}\right) = \delta_j \text{ for all } |j| < J\right\} = \frac{p}{2^{2J-1}} + O(J\sqrt{p}).$$

• This shows that the neighbouring values behave like independent random variables.

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Invariant L_4 Norm's Main Term

Corollary

For $0 < t \leq \frac{1}{2}$ and $0 \leq r \leq \frac{1}{2} - t$, we have

$$\|(F_p)_r^t\|_4^4 = \left\| \sum_{0 \leq \ell < tp} \left(\frac{\ell + \lfloor rp \rfloor}{p} \right) z^\ell \right\|_4^4 = (1+o(1)) \left(2 - \frac{4}{3} t \right) (tp)^2,$$ is independent of r.

- What we can say about these Littlewood polynomials if the main term of their L_4 norm is independent of the rotation r?
- If the Littlewood polynomials are ultra-flat, can we show that the main term of their L_4 norm is independent of the rotation r?

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Borwein and C. (2000)

For any non-principal and non-real character χ modulo p and $0 \le r < 1$, we have

$$\left\| \sum_{\ell=0}^{p-1} \chi(\ell + \lfloor rp \rfloor) z^{\ell} \right\|_4^4 = \frac{4}{3} p^2 + O(p^{3/2} \log^2 p).$$

THE END!