Using continued fractions to study metric Mahler measures

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- Lehmer's Conjecture: There exists $c > 0$ such that $m(\alpha) \geq c$ for all $\alpha \in \overline{\mathbb{Q}}^{\times} \setminus \overline{\mathbb{Q}}^{\times}_{\mathrm{tors}}$.

The Metric Mahler Measure

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Dubickas and Smyth (2001) defined the *metric Mahler measure* by

$$m_1(\alpha) = \inf\left\{\sum_{n=1}^{N} m(\alpha_n) : N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^\times,\ \alpha = \prod_{n=1}^{N} \alpha_n\right\}.$$

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- $m_1(\alpha\beta) \leq m_1(\alpha) + m_1(\beta)$ for all $\alpha, \beta \in \overline{\mathbb{Q}}^\times$.
- The map $(\alpha, \beta) \mapsto m_1(\alpha\beta^{-1})$ defines a metric on $\overline{\mathbb{Q}}^\times/\overline{\mathbb{Q}}^\times_{\mathrm{tors}}$ which induces the discrete topology if and only if Lehmer's conjecture is true.

The Parametrized Family of Metric Mahler Measures

Following the observations of Dubickas and Smyth, we defined the *t-metric Mahler measure* by

$$m_t(\alpha) = \inf\left\{ \left(\sum_{n=1}^{N} m(\alpha_n)^t \right)^{1/t} : N \in \mathbb{N},\ \alpha_n \in \overline{\mathbb{Q}}^{\times},\ \alpha = \prod_{n=1}^{N} \alpha_n \right\}.$$

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While this is surely a trivial change in the definition, it enables us to ask a new question.

Question 1.

What can we say about the map $t \mapsto m_t(\alpha)$ for fixed $\alpha \in \overline{\mathbb{Q}}^\times$?

Additional Definitions

- Any point $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in (\overline{\mathbb{Q}}^\times)^N$ satisfying $\alpha = \prod_{n=1}^N \alpha_n$ is called a *product representation of* α.

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- A point $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathcal{P}(\alpha)$ is said to *attain the infimum in* $m_t(\alpha)$ if

$$m_t(\alpha) = \left(\sum_{n=1}^{N} m(\alpha_n)^t \right)^{1/t}$$

The Parametrized Family of Metric Mahler Measures

Theorem 2 (S., 2014).

If $\alpha \in \overline{\mathbb{Q}}^\times$ then there exists a finite set $\mathcal{X} = \mathcal{X}(\alpha) \subseteq \mathcal{P}(\alpha)$ such that the infimum in $m_t(\alpha)$ is attained by a point in $\mathcal{X}$.

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1. **Good News:** This theorem is stronger than simply stating that the infimum in $m_t(\alpha)$ is attained for all α. Indeed, it shows that only a finite set of points is needed to attain that infimum as t varies over $\mathbb{R}^{+}$.

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1. **Good News:** This theorem is stronger than simply stating that the infimum in $m_t(\alpha)$ is attained for all α. Indeed, it shows that only a finite set of points is needed to attain that infimum as t varies over $\mathbb{R}^+$.
2. **Bad News:** The proof of the theorem provides no clue on how to determine $\mathcal{X}$. It cannot even provide an estimate on the size of $\mathcal{X}$ in terms of α.

Factorizations

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$$\left(\frac{r_1}{s_1}, \frac{r_2}{s_2}, \ldots, \frac{r_N}{s_N}\right) \in (\mathbb{Q}^\times)^N$$

of α is called a *factorization of* α if

1. $r_n/s_n > 0$ and $r_n/s_n \neq 1$ for all $1 \leq n \leq N$
2. $\gcd(r_m, s_n) = 1$ for all $1 \leq m, n \leq N$

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Each rational number has only finitely many factorizations.

Metric Mahler Measures of Rational Numbers

Theorem 3 (Jankauskas-S., 2012).

If $\alpha \neq 1$ is a positive rational number and $t > 0$ then there exists a factorization of α which attains the infimum in $m_t(\alpha)$.

This means we can use the set of all factorizations of α as $\mathcal{X}$ in our earlier theorem.

Metric Mahler measures of rational numbers

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$$\left(\frac{32}{27}\right) \quad \left(\frac{16}{27}, \frac{2}{1}\right) \quad \left(\frac{16}{9}, \frac{2}{3}\right) \quad \left(\frac{8}{9}, \frac{4}{3}\right)$$

$$\left(\frac{8}{9}, \frac{2}{3}, \frac{2}{1}\right) \quad \left(\frac{4}{3}, \frac{4}{3}, \frac{2}{3}\right) \quad \left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{1}\right) \quad \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{1}, \frac{2}{1}\right)$$

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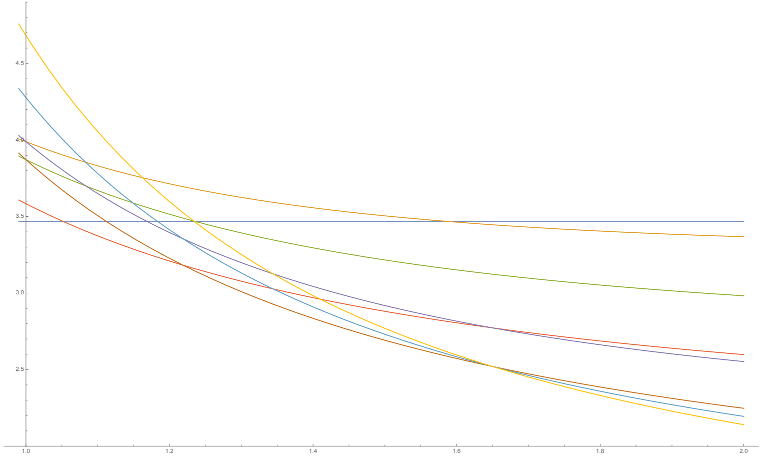
For your viewing benefit, we have trimmed the list of factorizations according to the following criterion. If $(\alpha_1, \dots, \alpha_N)$ and $(\beta_1, \dots, \beta_N)$ are factorizations of 32/27 with $m(\alpha_n) \leq m(\beta_n)$ for all n, then we may disregard $(\beta_1, \dots, \beta_N)$ in our calculations of $m_t(\alpha)$.

Metric Mahler Measures of Rational Numbers

The measure functions for factorizations of 32/27



Metric Mahler Measures of Rational Numbers

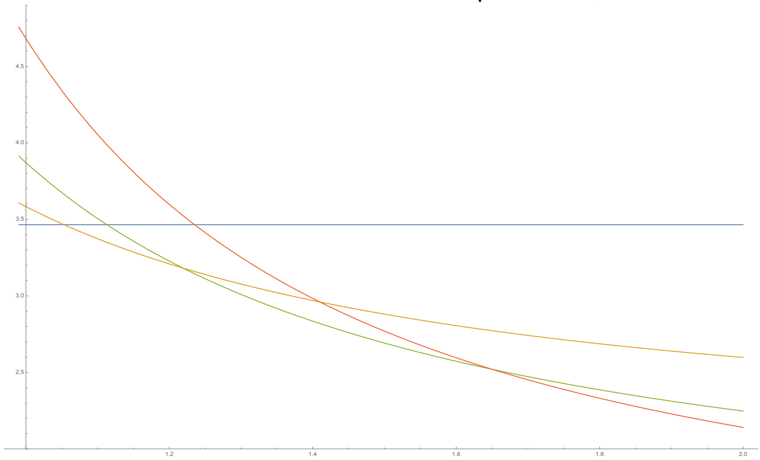
For every $t > 0$, the infimum in $m_t(32/27)$ is attained by a point in

$$\mathcal{Y} := \left\{ \left(\frac{32}{27} \right), \left(\frac{8}{9}, \frac{4}{3} \right), \left(\frac{4}{3}, \frac{4}{3}, \frac{2}{3} \right), \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{1}, \frac{2}{1} \right) \right\},$$

and moreover, no proper subset of $\mathcal{Y}$ also has this property.

Metric Mahler Measures of Rational Numbers

The measure functions for points in $\mathcal{Y}$



Infimum Attaining Factorizations

What is so special about these factorizations that attain the infimum in $m_t(32/27)$? Consider the two factorizations

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$$\left(\frac{8}{9}, \frac{4}{3}\right) \quad \text{and} \quad \left(\frac{16}{27}, \frac{2}{1}\right).$$

The terms in the first factorization are quite a bit closer to 1 than the terms in the second factorization. This *could* be relevant...

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- If p and q are distinct primes, then we observe that

$$e^{-\varepsilon} < \frac{p^a}{q^b} < e^{\varepsilon} \quad \Longleftrightarrow \quad \left| \frac{\log q}{\log p} - \frac{a}{b} \right| < \frac{\varepsilon}{b \log p}.$$

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  Roughly speaking, p^a/q^b is close to 1 if and only if a/b is close to $\log q/\log p$.
- If ξ is a positive real irrational number, we say that a/b, with $\gcd(a,b)=1$, is a *best rational approximation to* ξ if

$$\left|\frac{r}{s} - \xi\right| < \left|\frac{a}{b} - \xi\right| \quad \Longrightarrow \quad s > b.$$

Best Rational Approximations

Question 4.

Suppose a/b is a best rational approximation to $\log q/\log p$ and let $\alpha = p^a/q^b$. If

$$\left(\frac{p^{a_1}}{q^{b_1}}, \frac{p^{a_2}}{q^{b_2}}, \dots, \frac{p^{a_N}}{q^{b_N}}\right)$$

is a factorization of α attaining the infimum in $m_t(\alpha)$, must a_n/b_n be a best rational approximation to $\log q/\log p$ for all n?

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The answer is "no", but we're close: we need to use *upper* and *lower best approximations*.

Upper and Lower Best Approximations

Suppose that ξ is a positive real irrational number. A positive rational number a/b, with $\gcd(a, b) = 1$, is called an *upper best approximation to* ξ if

(i) $\xi < \frac{a}{b}$ and

(ii) If $\frac{r}{s} \in \mathbb{Q}$ is such that $\xi < \frac{r}{s} < \frac{a}{b}$ then $r > a$.

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Similarly, a/b is called a *lower best approximation to* ξ if

(i) $\frac{a}{b} < \xi$ and
(ii) If $\frac{r}{s} \in \mathbb{Q}$ is such that $\frac{a}{b} < \frac{r}{s} < \xi$ then $s > b$.

Infimum Attaining Points and Best Approximations

Theorem 5 (S., 2015).

Suppose p and q are distinct primes with $p < q$ and $\xi = \log q / \log p$. Assume that (a, b) is such that $\gcd(a, b) = 1$ and a/b is an upper or lower best approximation for ξ. If

$$\left(\frac{p^{a_1}}{q^{b_1}}, \frac{p^{a_2}}{q^{b_2}}, \ldots, \frac{p^{a_N}}{q^{b_N}} \right)$$

is a factorization attaining the infimum in $m_t(p^a/q^b)$ for some $t \neq 1$ then, for every $1 \leq n \leq N$, precisely one of the following holds:

(i) $a_n = 1$ *and* $b_n = 0$.

(ii) $(a_n, b_n) = 1$ *and* a_n/b_n *is an upper best approximation for* ξ.

(iii) $(a_n, b_n) = 1$ *and* a_n/b_n *is a lower best approximation for* ξ.

Listing Best Approximations

If $\xi > 1$ then there is a well-known strategy for listing the upper and lower best approximations to ξ.

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If $\xi > 1$ then there is a well-known strategy for listing the upper and lower best approximations to ξ. Supposing that ξ has continued fraction expansion

$$\xi = [x_0; x_1, x_2, x_3, \ldots],$$

then x_0 is a lower best approximation to ξ. Aside from this case, r/s is an upper or lower best approximation for ξ if and only if

$$\frac{r}{s} = [x_0; x_1, x_2, \ldots, x_{n-1}, x]$$

for some $n \in \mathbb{N}$ and $1 \leq x \leq x_n$.

Listing Best Approximations

Consider $\alpha = 2^a/3^b$ so that we set $\xi = \log 3/\log 2$. We obtain

$$\xi = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, \ldots].$$

Listing Best Approximations

Consider $\alpha = 2^a/3^b$ so that we set $\xi = \log 3/\log 2$. We obtain

$$\xi = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, \ldots].$$

The first several upper and lower best approximations are given as follows:

$$\text{Upper: } \frac{2}{1}, \frac{5}{3}, \frac{8}{5}, \frac{27}{17}, \frac{46}{29}, \frac{65}{41}$$

$$\text{Lower: } \frac{1}{1}, \frac{3}{2}, \frac{11}{7}, \frac{19}{12}, \frac{84}{53}$$

Listing Best Approximations

Let's return to our example of

$$\alpha = \frac{32}{27} = \frac{2^5}{3^3}.$$

In this case, $5/3$ is an upper best approximation for $\log 3/\log 2$, so that our main theorem applies. Hence, any factorization which attains the infimum in $m_t(\alpha)$ may only use rational numbers of the form

$$\frac{2}{1}, \frac{2}{3}, \frac{4}{3}, \frac{8}{9}, \frac{32}{27}.$$

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$$\frac{2}{1}, \ \frac{2}{3}, \ \frac{4}{3}, \ \frac{8}{9}, \ \frac{32}{27}.$$

The question of determining the infimum attaining factorizations of $32/27$ becomes a question about writing $(5,3)$ as a sum of points in

$$\{(1,0),(1,1),(2,1),(3,2),(5,3)\}.$$

Which Primes Create Good Examples?

- $\log 3/\log 2 = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, \ldots]$. The first several terms in its continued fraction expansion are small. This causes the heights of the upper and lower best approximations to grow quickly at first, which produces relatively few such points.

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- $\log 17/\log 13 = [1; 9, 1, 1, 1, 3, 1, 2, \ldots]$. The first several upper and lower best approximations are given by

$$\text{Upper: } \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}$$

$$\text{Lower: } \frac{1}{1}$$

The Best "Example"

The best example would give

$$\frac{\log q}{\log p} = [1; 1, 1, 1, 1, \ldots]$$

but this is not possible since the right hand side equals $(1+\sqrt{5})/2$.

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Lemma 6.

If $n \in \mathbb{N}$ then there exists primes p and q such that

$$\frac{\log q}{\log p} = [1; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}, x_{n+1,n+2}, \ldots]$$

A Fancy Example

If we take $p = 31$ and $q = 257$ we obtain

$$\frac{\log 257}{\log 31} = [1; 1, 1, 1, 1, 1, 1, 10, 3, 1, 1, 1, 5, \ldots].$$

In particular, we find that $21/13$ is a lower best approximation to $\log 257/\log 31$. Therefore,

$$\alpha = \frac{31^{21}}{257^{13}}$$

satisfies the required properties to apply our main theorem.

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$$\frac{\log 257}{\log 31} = [1; 1, 1, 1, 1, 1, 1, 10, 3, 1, 1, 1, 5, \ldots].$$

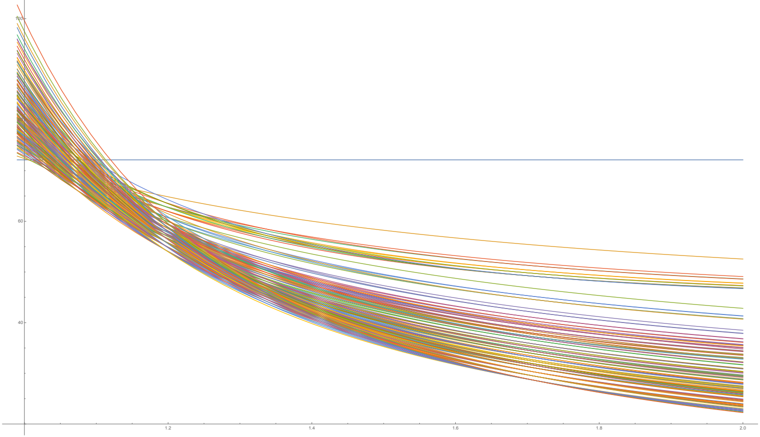
In particular, we find that $21/13$ is a lower best approximation to $\log 257/\log 31$. Therefore,

$$\alpha = \frac{31^{21}}{257^{13}}$$

satisfies the required properties to apply our main theorem. There are 139 ways to write $(21, 13)$ as a sum of upper and lower best approximations to $\log 257/\log 31$ each giving rise to a corresponding measure function.

A Fancy Example

The 139 relevant measure functions for $\frac{31^{21}}{257^{13}}$



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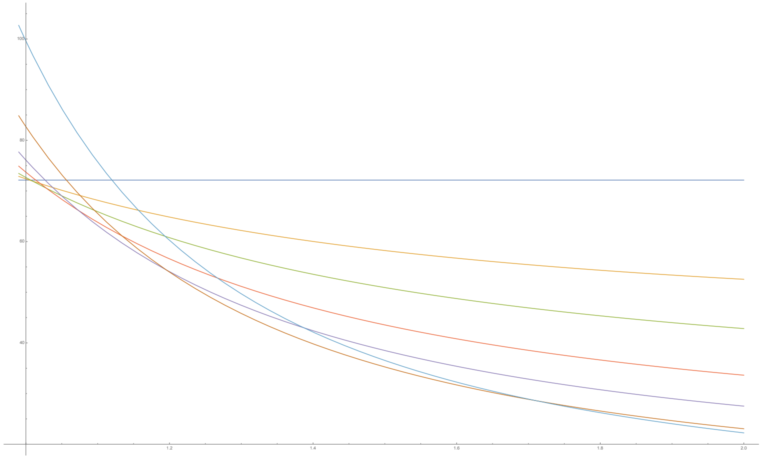
In this case, only seven of these measure functions are needed in order to attain the infimum in $m_t(31^{21}/257^{13})$.



The End