Salem numbers with negative trace

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Salem numbers

Totally positive algebraic numbers

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Formulation as an optimisation problem

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Definition

- A Salem number is an algebraic integer $\tau > 1$ of degree $2d \geq 4$, all of whose conjugates, apart from τ and τ^{-1}, have modulus 1. Its minimal polynomial $Q = z^{2d} + b_1 z^{2d-1} + \cdots$ is reciprocal: $z^{2d} Q(1/z) = Q(z)$.

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- Lehmer's $L(x) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.

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- **Theorem** J. McKee and C. J. Smyth (2005). *For every negative integer $-T$, there is a Salem number of trace $-T$ and degree at most* $\exp\exp(22 + 4TlogT)$.

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- **New:** A Salem number with trace -3 and degree 34.

Relation between Salem numbers and totally positive algebraic integers.

- By $x = z + 1/z + 2$ we get $P \in \mathbb{Z}[x]$, a totally positive polynomial with degree d whose all roots but one are < 4. Its trace is $2d - b_1$.

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- For fixed b_1 find minimal possible d.
- For b_1 and d fixed find P.

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- **The Schur-Siegel-Smyth trace problem:** is 2 a limit point of the set of all such rational numbers?

The method of explicit auxiliary functions

- This method was introduced by C.J. Smyth(1984). Define f on $(0, \infty)$:

$$f(x) = x - \sum_{1 \leq j \leq J} e_j \log(|Q_j(x)| \geq m,$$

with $e_j > 0$ and $Q_j \in \mathbb{Z}[x]^*$.

The method of explicit auxiliary functions

- Let α be a totally positive algebraic integer with conjugates $\alpha_1, \ldots, \alpha_d$ and P its minimal polynomial which does not divide any Q_j, so $\mathrm{Res}(P, Q_j) \in \mathbb{Z}^*$, then summing on α_i

$$\mathrm{trace}(\alpha) \geq md + \sum_{1 \leq j \leq J} e_j \log(|\mathrm{Res}(P, Q_j)|).$$

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- Smyth(1984) $\rightarrow m = 1.7719$.
- Heuristic method $\rightarrow$ 1.784109 Aguirre and Peral (2006).

The integer transfinite diameter

- Let K be a compact subset of $\mathbb{C}$, the *integer transfinite diameter of K* is defined as

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n \geq 1 \\ n \to \infty}} \inf_{\substack{P \in \mathbb{Z}[z] \\ \deg(P) = n}} |P|_{\infty,K}^{\frac{1}{n}},$$

where $|P|_{\infty,K} = \sup_{z \in K} |P(z)|$.

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- Pritsker $0.4213 < t_{\mathbb{Z}}[0,1] < 0.422685$ Flammang (2013) . (2008)

The φ-*integer transfinite diameter*

- F. Amoroso(1993): Let K be a compact subset of $\mathbb{C}$. If φ is a positive function defined on K, the φ-*integer transfinite diameter of* K is defined as

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \to \infty}} \inf_{\substack{P \in \mathbb{Z}[z] \\ \deg(P) = n}} \sup_{z \in K} \left(|P(z)|^{\frac{1}{n}} \varphi(z) \right).$$

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- Extended to the case : $K = (0, \infty)$ and $\varphi(z) = e^{-z}$.

Construction of explicit auxiliary functions

- $c_j \in \mathbb{Q}$, $f(x) = x - \frac{t}{r}\log|Q(x)| \geq m, r = degree(Q)$.
  If $t = 1$, $exp(-f(x))$ becomes

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$$\text{minimize} \quad \max_{1\leq k\leq K} |Q(x_k)| e^{-rx_k}.$$

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- With a recursive version, Flammang(2009): 1.792519.

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- Serre's upper bound : 1.8983021.
- Anyway, the auxiliary functions are used to give a lower bound for d and an upper bound for the roots of P.

Find totally positive polynomials with degree 17 and trace 31

- If trace$(Q) = -3$, Flammang's function implies $d \geq 15$. Search for $d = 15$ and $d = 16$ gives no Salem number.

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- Remember that the trace of P is equal to 31.

Formulation of the problem

- Find $P = x^{17} + \sum_{i=1}^{17} a_i x^{17-i}$. With
  - $a_1, a_2, \ldots, a_{17}$ integers, $a_1 = -31$
  - 16 real roots in $(0, 4)$, one in $(4, 6.69)$ (*)

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- Method
  If we suppose that $P(0) < 0$, $P(\beta_1) > 0$, $P(\beta_2) < 0, \ldots,$ $P(\beta_{16} = 4) < 0$, $P(6.69) > 0$ for $0 < \beta_1 < \ldots < \beta_{16}$ in $(0, 6.69)$, then the condition (*) is satisfied for P.

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- The numbers β_i are a set of “separators” for the roots of P.

Formulation as a system of linear inequalities

- ↬ <u>New formulation</u>
  For hundreds of thousands $(\beta_1, \beta_2, \ldots, \ \beta_{16})$,

$$\text{minimize} \sum_{i=1}^{17} a_i, \text{ such that } \begin{cases} p(0) < 0, \\ p(\beta_1) > 0, \\ \vdots \\ p(\beta_{16}) < 0, \\ p(6.69) > 0, \\ (a_1, \cdots, a_{17}) \in \mathbb{Z}^d. \end{cases}$$

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- “Find a needle in a haystack”
- “Trouver une aiguille dans une botte de foin”

The result

- $x^{17} - 31x^{16} + 433x^{15} - 3608x^{14} + 20013x^{13} - 78079x^{12} + 220717x^{11} - 458940x^{10} + 705459x^{9} - 799257x^{8} + 660596x^{7} - 391294x^{6} + 161786x^{5} - 44982x^{4} + 7979x^{3} - 837x^{2} + 46x - 1$

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- The Salem number is $\tau = 2.7616448085\ldots$
- $z^{34} + 3z^{33} + 2z^{32} - 10z^{31} - 40z^{30} - 89z^{29} - 149z^{28} - 208z^{27} - 257z^{26} - 293z^{25} - 315z^{24} - 322z^{23} - 311z^{22} - 281z^{21} - 237z^{20} - 191z^{19} - 156z^{18} - 143z^{17} - 156z^{16} - 191z^{15} - 237z^{14} - 281z^{13} - 311z^{12} - 322z^{11} - 315z^{10} - 293z^{9} - 257z^{8} - 208z^{7} - 149z^{6} - 89z^{5} - 40z^{4} - 10z^{3} + 2z^{2} + 3z + 1.$

More auxiliary functions

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- $x \to \log(1 + x)$: $L(\alpha)$ is the absolute length of α.
- $L(\alpha) \geq 2.365827\ldots$ by Flammang(2013) with 88 polynomials, 1224 roots. This is close to the upper bound $2.373605\ldots$ deduced from Pritsker's result on integer transfinite diameter of $[0, 1]$ by potential theory.

Question

- It would be interesting to explain why the situation of the auxiliary functions for the **trace** and for the **length** are so different!