Means of algebraic numbers

Igor E. Pritsker

Oklahoma State University

http://www.math.okstate.edu/~igor/

Arithmetic means of algebraic numbers in the unit disk

Schur considered polynomials $P_n(z) = a_n \prod_{k=1}^{n}(z-\alpha_k) \in \mathbb{Z}[z]$ with simple zeros in the unit disk, and the means $A_n := \frac{1}{n}\sum_{k=1}^{n}\alpha_k$.

Schur, 1918: If $|a_n| \le M,\ n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} |A_n| \le 1 - \sqrt{e}/2 \approx 0.1756.$$

For $\tau_n := \frac{1}{n}\sum_{k=1}^{n}\delta_{\alpha_k}$ and $d\mu_{\mathbb{T}}(e^{it}) := \frac{1}{2\pi}dt$, $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ is defined by

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\alpha_k) = \lim_{n\to\infty}\int f(z)\,d\tau_n(z) = \int f(z)\,d\mu_{\mathbb{T}}(z),\ f \in C_0(\mathbb{C}).$$

Theorem

Let $P_n(z) = a_n \prod_{k=1}^{n}(z-\alpha_k) \in \mathbb{Z}[z]$ have simple zeros. If $\lim_{n\to\infty}|a_n|^{1/n} = 1$ and $|\alpha_k| \le 1,\ k = 1,\ldots,n$, then $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\alpha_k^m = \lim_{n\to\infty}\int z^m\,d\tau_n(z) = \int z^m\,d\mu_{\mathbb{T}}(z) = 0,\ m \in \mathbb{N}.$$

Arithmetic means of algebraic numbers in the unit disk

Schur considered polynomials $P_n(z) = a_n \prod_{k=1}^n (z-\alpha_k) \in \mathbb{Z}[z]$ with simple zeros in the unit disk, and the means $A_n := \frac{1}{n}\sum_{k=1}^n \alpha_k$.

Schur, 1918: If $|a_n| \le M,\ n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} |A_n| \le 1 - \sqrt{e}/2 \approx 0.1756.$$

For $\tau_n := \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}$ and $d\mu_{\mathbb{T}}(e^{it}) := \frac{1}{2\pi}dt$, $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ is defined by

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n f(\alpha_k) = \lim_{n\to\infty} \int f(z)\, d\tau_n(z) = \int f(z)\, d\mu_{\mathbb{T}}(z),\ \ f \in C_0(\mathbb{C}).$$

Theorem

Let $P_n(z) = a_n \prod_{k=1}^n (z-\alpha_k) \in \mathbb{Z}[z]$ have simple zeros. If $\lim_{n\to\infty} |a_n|^{1/n} = 1$ and $|\alpha_k| \le 1,\ k = 1, \ldots, n$, then $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \alpha_k^m = \lim_{n\to\infty} \int z^m\, d\tau_n(z) = \int z^m\, d\mu_{\mathbb{T}}(z) = 0,\ \ m \in \mathbb{N}.$$

Arithmetic means of algebraic numbers in the unit disk

Schur considered polynomials $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros in the unit disk, and the means $A_n := \frac{1}{n} \sum_{k=1}^n \alpha_k$.

Schur, 1918: If $|a_n| \le M, \ n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} |A_n| \le 1 - \sqrt{e}/2 \approx 0.1756.$$

For $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$ and $d\mu_{\mathbb{T}}(e^{it}) := \frac{1}{2\pi} dt$, $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ is defined by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n f(\alpha_k) = \lim_{n\to\infty} \int f(z)\, d\tau_n(z) = \int f(z)\, d\mu_{\mathbb{T}}(z), \ f \in C_0(\mathbb{C}).$$

Theorem

Let $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ have simple zeros. If $\lim_{n\to\infty} |a_n|^{1/n} = 1$ and $|\alpha_k| \le 1, \ k = 1, \ldots, n$, then $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \alpha_k^m = \lim_{n\to\infty} \int z^m\, d\tau_n(z) = \int z^m\, d\mu_{\mathbb{T}}(z) = 0, \ m \in \mathbb{N}.$$

Rate of convergence via Mahler's measure

Let polynomials $P_n(z) = a_n \prod_{k=1}^{n}(z-\alpha_k) \in \mathbb{Z}[z]$ have simple zeros, and define $M(P_n) = |a_n| \prod_{|\alpha_k|>1} |\alpha_k|$.

Theorem

If $|f(z) - f(t)| \le A|z-t|,\ z,t \in \mathbb{C}$, *with* $\mathrm{supp}(f) \subset \{|z| \le R\}$, *then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\alpha_k) - \int f\, d\mu_{\mathbb{T}} \right| \le A(2R+1)\sqrt{\frac{\log \max(n, M(P_n))}{n}},\ \ n \ge 55.$$

Corollary

If $|a_n| \le M$ *and* $|\alpha_k| \le 1,\ k = 1, \ldots, n$, *then* $|A_n| \le 8\sqrt{\log n/n}$ *and* $\max_{\mathbb{D}} |P_n| \le e^{c\sqrt{n}\log n}$ *for* $c > 0$ *and* $n \ge \max(M, 55)$.

For sharpness, let p_m be the mth prime number and consider

$$P_n(z) := \prod_{m=1}^{k} \frac{z^{p_m} - 1}{z - 1} = \prod_{m=1}^{k} \sum_{j=0}^{p_m - 1} z^j, \quad k \in \mathbb{N}.$$

Rate of convergence via Mahler's measure

Let polynomials $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ have simple zeros, and define $M(P_n) = |a_n| \prod_{|\alpha_k|>1} |\alpha_k|$.

Theorem

If $|f(z) - f(t)| \le A|z - t|,\ z, t \in \mathbb{C}$, *with* $\operatorname{supp}(f) \subset \{|z| \le R\}$, *then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\alpha_k) - \int f\, d\mu_{\mathbb{T}} \right| \le A(2R+1)\sqrt{\frac{\log \max(n, M(P_n))}{n}},\ \ n \ge 55.$$

Corollary

If $|a_n| \le M$ *and* $|\alpha_k| \le 1,\ k = 1, \ldots, n$, *then* $|A_n| \le 8\sqrt{\log n/n}$ *and* $\max_{\mathbb{D}} |P_n| \le e^{c\sqrt{n}\log n}$ *for* $c > 0$ *and* $n \ge \max(M, 55)$.

For sharpness, let p_m be the mth prime number and consider

$$P_n(z) := \prod_{m=1}^{k} \frac{z^{p_m} - 1}{z - 1} = \prod_{m=1}^{k} \sum_{j=0}^{p_m - 1} z^j, \quad k \in \mathbb{N}.$$

Rate of convergence via Mahler's measure

Let polynomials $P_n(z) = a_n \prod_{k=1}^{n}(z - \alpha_k) \in \mathbb{Z}[z]$ have simple zeros, and define $M(P_n) = |a_n| \prod_{|\alpha_k|>1} |\alpha_k|$.

Theorem

If $|f(z) - f(t)| \le A|z - t|,\ z, t \in \mathbb{C}$, *with* $\operatorname{supp}(f) \subset \{|z| \le R\}$, *then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\alpha_k) - \int f\, d\mu_{\mathbb{T}} \right| \le A(2R+1)\sqrt{\frac{\log \max(n, M(P_n))}{n}},\ \ n \ge 55.$$

Corollary

If $|a_n| \le M$ *and* $|\alpha_k| \le 1,\ k = 1, \ldots, n$, *then* $|A_n| \le 8\sqrt{\log n/n}$ *and* $\max_{\mathbb{D}} |P_n| \le e^{c\sqrt{n}\log n}$ *for* $c > 0$ *and* $n \ge \max(M, 55)$.

For sharpness, let p_m be the mth prime number and consider

$$P_n(z) := \prod_{m=1}^{k} \frac{z^{p_m} - 1}{z - 1} = \prod_{m=1}^{k} \sum_{j=0}^{p_m - 1} z^j, \quad k \in \mathbb{N}.$$

Rate of convergence via Mahler's measure

Let polynomials $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ have simple zeros, and define $M(P_n) = |a_n| \prod_{|\alpha_k|>1} |\alpha_k|$.

Theorem

If $|f(z) - f(t)| \le A|z - t|, \ z, t \in \mathbb{C}$, *with* $\text{supp}(f) \subset \{|z| \le R\}$, *then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(\alpha_k) - \int f \, d\mu_{\mathbb{T}} \right| \le A(2R+1)\sqrt{\frac{\log \max(n, M(P_n))}{n}}, \quad n \ge 55.$$

Corollary

If $|a_n| \le M$ *and* $|\alpha_k| \le 1, \ k = 1, \ldots, n$, *then* $|A_n| \le 8\sqrt{\log n/n}$ *and* $\max_{\mathbb{D}} |P_n| \le e^{c\sqrt{n}\log n}$ *for* $c > 0$ *and* $n \ge \max(M, 55)$.

For sharpness, let p_m be the mth prime number and consider

$$P_n(z) := \prod_{m=1}^{k} \frac{z^{p_m} - 1}{z - 1} = \prod_{m=1}^{k} \sum_{j=0}^{p_m - 1} z^j, \quad k \in \mathbb{N}.$$

Problem: Close the gap

Let p_m be the mth prime number and consider

$$P_n(z) := \prod_{m=1}^{k} \frac{z^{p_m} - 1}{z - 1} = \prod_{m=1}^{k} \sum_{j=0}^{p_m - 1} z^j, \quad k \in \mathbb{N}.$$

Using PNT (with remainder), one can show for this P_n that

$$|A_n| \geq \frac{c_1}{\sqrt{n \log n}} \quad \left(\text{vs. } \leq 8\sqrt{\frac{\log n}{n}} \right)$$

and

$$\max_{\mathbb{D}} |P_n| = P_n(1) = \prod_{m=1}^{k} p_m \geq e^{c_2 \sqrt{n \log n}} \quad \left(\text{vs. } \leq e^{c_3 \sqrt{n} \log n} \right).$$

Schur-Siegel-Smyth trace problem

Let $P_n(x) = \prod_{k=1}^n (x - \alpha_k) = \sum_{k=0}^n a_k x^k \in \mathbb{Z}[x]$ be irreducible. Define the trace $\mathrm{tr}(\alpha) := \sum_{k=1}^n \alpha_k = (-1)^n a_{n-1}$.

Theorem (Schur, 1918)

If $\alpha_k > 0,\ k = 1, \ldots, n$, (totally positive) then $\mathrm{tr}(\alpha)/n > \sqrt{e} > 1.6487$, with finitely many explicit exceptions.

Problem

Find $\ell := \liminf\{\mathrm{tr}(\alpha)/n : \alpha_k > 0$ of degree $n\}$.

Schur and Siegel observed that $\ell \le 2$. Indeed, $2T_p(x/2-1)/(x-2)$ is an irreducible monic polynomial for any odd prime p. Hence it's root $4\cos^2(\pi/(4p))$ is an algebraic integer of degree $p-1$ and trace $2p-2$.

Siegel, 1945: $\ell > 1.7336105$ by improving the AGM inequality $\frac{1}{n}\sum_{k=1}^n \alpha_k \ge \left(\prod_{k=1}^n \alpha_k\right)^{1/n} = |a_0|^{1/n} \ge 1$.

Schur-Siegel-Smyth trace problem

Let $P_n(x) = \prod_{k=1}^n (x - \alpha_k) = \sum_{k=0}^n a_k x^k \in \mathbb{Z}[x]$ be irreducible. Define the trace $\mathrm{tr}(\alpha) := \sum_{k=1}^n \alpha_k = (-1)^n a_{n-1}$.

Theorem (Schur, 1918)

If $\alpha_k > 0,\ k = 1, \ldots, n$, *(totally positive) then* $\mathrm{tr}(\alpha)/n > \sqrt{e} > 1.6487$, *with finitely many explicit exceptions.*

Problem

Find $\ell := \liminf\{\mathrm{tr}(\alpha)/n : \alpha_k > 0$ *of degree* $n\}$.

Schur and Siegel observed that $\ell \le 2$**.** Indeed, $2T_p(x/2-1)/(x-2)$ is an irreducible monic polynomial for any odd prime p. Hence it's root $4\cos^2(\pi/(4p))$ is an algebraic integer of degree $p-1$ and trace $2p-2$.

Siegel, 1945: $\ell > 1.7336105$ by improving the AGM inequality $\frac{1}{n}\sum_{k=1}^n \alpha_k \ge \left(\prod_{k=1}^n \alpha_k\right)^{1/n} = |a_0|^{1/n} \ge 1.$

Schur-Siegel-Smyth trace problem

Let $P_n(x) = \prod_{k=1}^n (x - \alpha_k) = \sum_{k=0}^n a_k x^k \in \mathbb{Z}[x]$ be irreducible. Define the trace $\mathrm{tr}(\alpha) := \sum_{k=1}^n \alpha_k = (-1)^n a_{n-1}$.

Theorem (Schur, 1918)

If $\alpha_k > 0,\ k = 1, \ldots, n$, (totally positive) then $\mathrm{tr}(\alpha)/n > \sqrt{e} > 1.6487$, with finitely many explicit exceptions.

Problem

Find $\ell := \liminf\{\mathrm{tr}(\alpha)/n : \alpha_k > 0$ of degree $n\}$.

Schur and Siegel observed that $\ell \le 2$. Indeed, $2T_p(x/2-1)/(x-2)$ is an irreducible monic polynomial for any odd prime p. Hence it's root $4\cos^2(\pi/(4p))$ is an algebraic integer of degree $p-1$ and trace $2p-2$.

Siegel, 1945: $\ell > 1.7336105$ by improving the AGM inequality

$$\frac{1}{n}\sum_{k=1}^n \alpha_k \ge \left(\prod_{k=1}^n \alpha_k\right)^{1/n} = |a_0|^{1/n} \ge 1.$$

Schur-Siegel-Smyth trace problem

Let $P_n(x) = \prod_{k=1}^n (x - \alpha_k) = \sum_{k=0}^n a_k x^k \in \mathbb{Z}[x]$ be irreducible. Define the trace $\mathrm{tr}(\alpha) := \sum_{k=1}^n \alpha_k = (-1)^n a_{n-1}$.

Theorem (Schur, 1918)

If $\alpha_k > 0$, $k = 1, \ldots, n$, (totally positive) then $\mathrm{tr}(\alpha)/n > \sqrt{e} > 1.6487$, with finitely many explicit exceptions.

Problem

Find $\ell := \liminf\{\mathrm{tr}(\alpha)/n : \alpha_k > 0$ of degree $n\}$.

Schur and Siegel observed that $\ell \le 2$. Indeed, $2T_p(x/2-1)/(x-2)$ is an irreducible monic polynomial for any odd prime p. Hence it's root $4\cos^2(\pi/(4p))$ is an algebraic integer of degree $p-1$ and trace $2p-2$.

Siegel, 1945: $\ell > 1.7336105$ by improving the AGM inequality $\frac{1}{n}\sum_{k=1}^n \alpha_k \ge \left(\prod_{k=1}^n \alpha_k\right)^{1/n} = |a_0|^{1/n} \ge 1.$

Schur-Siegel-Smyth trace problem

Let $P_n(x) = \prod_{k=1}^{n}(x-\alpha_k) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x]$ be irreducible. Define the trace $\mathrm{tr}(\alpha) := \sum_{k=1}^{n} \alpha_k = (-1)^n a_{n-1}$.

Theorem (Schur, 1918)

If $\alpha_k > 0,\ k = 1, \ldots, n$, (totally positive) then $\mathrm{tr}(\alpha)/n > \sqrt{e} > 1.6487$, with finitely many explicit exceptions.

Problem

Find $\ell := \liminf\{\mathrm{tr}(\alpha)/n : \alpha_k > 0$ of degree $n\}$.

Schur and Siegel observed that $\ell \leq 2$. Indeed, $2T_p(x/2-1)/(x-2)$ is an irreducible monic polynomial for any odd prime p. Hence it's root $4\cos^2(\pi/(4p))$ is an algebraic integer of degree $p-1$ and trace $2p-2$.
Siegel, 1945: $\ell > 1.7336105$ by improving the AGM inequality
$\frac{1}{n}\sum_{k=1}^{n} \alpha_k \geq \left(\prod_{k=1}^{n} \alpha_k\right)^{1/n} = |a_0|^{1/n} \geq 1.$

Smyth's method of "auxiliary polynomials"

Let $P_j \in \mathbb{Z}[z]$ be irreducible, and let $\lambda_j > 0,\ j = 1, \ldots, m$. One needs $x - \sum_{j=1}^m \lambda_j \log|P_j(x)| \geq 1.7719$ for all $x > 0$, to show

$$\frac{\mathrm{tr}(\alpha)}{n} \geq 1.7719 + \frac{1}{n} \sum_{j=1}^{m} \lambda_j \log|Res(P, P_j)|$$

for any totally positive α of degree n with minimal polynomial P.

Smyth, 1984: $\ell > 1.7719$
Flammang, Grandcolas and Rhin, 1997: $\ell > 1.7735$
McKee and Smyth, 2004: $\ell > 1.7783$
Aguirre, Bilbao and Peral, 2006: $\ell > 1.7800$
Aguirre and Peral, 2008: $\ell > 1.7841$
Flammang, 2009: $\ell > 1.78702$
McKee, 2011: $\ell > 1.78839$
Liang and Wu, 2011: $\ell > 1.79193$

Smyth and Serre, 1999: Any lower bound found by this method cannot be larger than 1.8983021.

Smyth's method of "auxiliary polynomials"

Let $P_j \in \mathbb{Z}[z]$ be irreducible, and let $\lambda_j > 0,\ j = 1, \ldots, m$. One needs $x - \sum_{j=1}^m \lambda_j \log|P_j(x)| \ge 1.7719$ for all $x > 0$, to show

$$\frac{\operatorname{tr}(\alpha)}{n} \ge 1.7719 + \frac{1}{n}\sum_{j=1}^m \lambda_j \log|Res(P, P_j)|$$

for any totally positive α of degree n with minimal polynomial P.

Smyth, 1984: $\ell > 1.7719$
Flammang, Grandcolas and Rhin, 1997: $\ell > 1.7735$
McKee and Smyth, 2004: $\ell > 1.7783$
Aguirre, Bilbao and Peral, 2006: $\ell > 1.7800$
Aguirre and Peral, 2008: $\ell > 1.7841$
Flammang, 2009: $\ell > 1.78702$
McKee, 2011: $\ell > 1.78839$
Liang and Wu, 2011: $\ell > 1.79193$

Smyth and Serre, 1999: Any lower bound found by this method cannot be larger than 1.8983021.

Smyth's method of "auxiliary polynomials"

Let $P_j \in \mathbb{Z}[z]$ be irreducible, and let $\lambda_j > 0,\ j = 1, \ldots, m$. One needs $x - \sum_{j=1}^m \lambda_j \log|P_j(x)| \geq 1.7719$ for all $x > 0$, to show

$$\frac{\mathrm{tr}(\alpha)}{n} \geq 1.7719 + \frac{1}{n}\sum_{j=1}^{m} \lambda_j \log|Res(P, P_j)|$$

for any totally positive α of degree n with minimal polynomial P.

Smyth, 1984: $\ell > 1.7719$
Flammang, Grandcolas and Rhin, 1997: $\ell > 1.7735$
McKee and Smyth, 2004: $\ell > 1.7783$
Aguirre, Bilbao and Peral, 2006: $\ell > 1.7800$
Aguirre and Peral, 2008: $\ell > 1.7841$
Flammang, 2009: $\ell > 1.78702$
McKee, 2011: $\ell > 1.78839$
Liang and Wu, 2011: $\ell > 1.79193$

Smyth and Serre, 1999: Any lower bound found by this method cannot be larger than 1.8983021.

Smyth's method of "auxiliary polynomials"

Let $P_j \in \mathbb{Z}[z]$ be irreducible, and let $\lambda_j > 0,\ j = 1, \ldots, m$. One needs $x - \sum_{j=1}^m \lambda_j \log|P_j(x)| \geq 1.7719$ for all $x > 0$, to show

$$\frac{\mathrm{tr}(\alpha)}{n} \geq 1.7719 + \frac{1}{n}\sum_{j=1}^{m} \lambda_j \log|Res(P, P_j)|$$

for any totally positive α of degree n with minimal polynomial P.

Smyth, 1984: $\ell > 1.7719$
Flammang, Grandcolas and Rhin, 1997: $\ell > 1.7735$
McKee and Smyth, 2004: $\ell > 1.7783$
Aguirre, Bilbao and Peral, 2006: $\ell > 1.7800$
Aguirre and Peral, 2008: $\ell > 1.7841$
Flammang, 2009: $\ell > 1.78702$
McKee, 2011: $\ell > 1.78839$
Liang and Wu, 2011: $\ell > 1.79193$

Smyth and Serre, 1999: Any lower bound found by this method cannot be larger than 1.8983021.

Mahler measure and equidistribution

Given a compact set $E \subset \mathbb{C}$, $\text{cap}(E) = 1$, let Ω_E be the unbounded component of $\overline{\mathbb{C}} \setminus E$, and let $g_E(z, \infty)$ be the Green function of Ω_E. For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros, define $m_E(P_n) := \log|a_n| + \sum_{\alpha_k \in \Omega_E} g_E(\alpha_k, \infty)$. Let μ_E be the equilibrium measure of E, and let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$.

Theorem (Bilu, Bombieri, Rumely, Favre and Rivera-Letelier, ...)

If $\lim_{n\to\infty} m_E(P_n)/n = 0$ then $\tau_n \xrightarrow{*} \mu_E$ as $n \to \infty$.

Ex. 1 (Bilu) If $E = D = \{|z| \le 1\}$ then $g_D(z, \infty) = \log|z|$, $m_D(P_n) = \log|a_n| + \sum_{|\alpha_k|>1} \log|\alpha_k|$ and $d\mu_D(e^{i\theta}) = d\theta/(2\pi)$.

Ex. 2 If $E = [0, 4]$ then $g_{[0,4]}(z, \infty) = \log|(z - 2 + \sqrt{z^2 - 4z})/2|$ and $d\mu_{[0,4]}(x) = dx/(\pi\sqrt{x(4-x)}),\ x \in [0, 4]$.

For $P_n(z) = 2T_n(z/2 - 1)$, we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4-x)})$ and

$$\lim_{n\to\infty} \frac{\text{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Mahler measure and equidistribution

Given a compact set $E \subset \mathbb{C}$, $\mathrm{cap}(E) = 1$, let Ω_E be the unbounded component of $\overline{\mathbb{C}} \setminus E$, and let $g_E(z, \infty)$ be the Green function of Ω_E. For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros, define $m_E(P_n) := \log |a_n| + \sum_{\alpha_k \in \Omega_E} g_E(\alpha_k, \infty)$. Let μ_E be the equilibrium measure of E, and let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$.

Theorem (Bilu, Bombieri, Rumely, Favre and Rivera-Letelier, . . .)

If $\lim_{n\to\infty} m_E(P_n)/n = 0$ *then* $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$.

Ex. 1 (Bilu) If $E = D = \{|z| \le 1\}$ then $g_D(z, \infty) = \log |z|$, $m_D(P_n) = \log |a_n| + \sum_{|\alpha_k|>1} \log |\alpha_k|$ and $d\mu_D(e^{i\theta}) = d\theta/(2\pi)$.

Ex. 2 If $E = [0, 4]$ then $g_{[0,4]}(z, \infty) = \log |(z - 2 + \sqrt{z^2 - 4z})/2|$ and $d\mu_{[0,4]}(x) = dx/(\pi\sqrt{x(4 - x)}),\ x \in [0, 4]$.

For $P_n(z) = 2T_n(z/2 - 1)$, we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4 - x)})$ and

$$\lim_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4 - x)}} = 2.$$

Mahler measure and equidistribution

Given a compact set $E \subset \mathbb{C}$, $\text{cap}(E) = 1$, let Ω_E be the unbounded component of $\overline{\mathbb{C}} \setminus E$, and let $g_E(z, \infty)$ be the Green function of Ω_E. For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros, define $m_E(P_n) := \log |a_n| + \sum_{\alpha_k \in \Omega_E} g_E(\alpha_k, \infty)$. Let μ_E be the equilibrium measure of E, and let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$.

Theorem (Bilu, Bombieri, Rumely, Favre and Rivera-Letelier, . . .)

If $\lim_{n\to\infty} m_E(P_n)/n = 0$ *then* $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$.

Ex. 1 (Bilu) If $E = D = \{|z| \le 1\}$ then $g_D(z, \infty) = \log |z|$, $m_D(P_n) = \log |a_n| + \sum_{|\alpha_k|>1} \log |\alpha_k|$ and $d\mu_D(e^{i\theta}) = d\theta/(2\pi)$.
Ex. 2 If $E = [0, 4]$ then $g_{[0,4]}(z, \infty) = \log |(z - 2 + \sqrt{z^2 - 4z})/2|$ and $d\mu_{[0,4]}(x) = dx/(\pi\sqrt{x(4 - x)}),\ x \in [0, 4]$.
For $P_n(z) = 2T_n(z/2 - 1)$, we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4 - x)})$ and

$$\lim_{n\to\infty} \frac{\text{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4 - x)}} = 2.$$

Mahler measure and equidistribution

Given a compact set $E \subset \mathbb{C}$, $\operatorname{cap}(E) = 1$, let Ω_E be the unbounded component of $\overline{\mathbb{C}} \setminus E$, and let $g_E(z, \infty)$ be the Green function of Ω_E. For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros, define $m_E(P_n) := \log|a_n| + \sum_{\alpha_k \in \Omega_E} g_E(\alpha_k, \infty)$. Let μ_E be the equilibrium measure of E, and let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$.

Theorem (Bilu, Bombieri, Rumely, Favre and Rivera-Letelier, ...)

If $\lim_{n\to\infty} m_E(P_n)/n = 0$ *then* $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$.

Ex. 1 (Bilu) If $E = D = \{|z| \le 1\}$ then $g_D(z, \infty) = \log|z|$, $m_D(P_n) = \log|a_n| + \sum_{|\alpha_k|>1} \log|\alpha_k|$ and $d\mu_D(e^{i\theta}) = d\theta/(2\pi)$.

Ex. 2 If $E = [0, 4]$ then $g_{[0,4]}(z, \infty) = \log|(z - 2 + \sqrt{z^2 - 4z})/2|$ and $d\mu_{[0,4]}(x) = dx/(\pi\sqrt{x(4-x)}),\ x \in [0, 4]$.

For $P_n(z) = 2T_n(z/2 - 1)$, we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4-x)})$ and

$$\lim_{n\to\infty} \frac{\operatorname{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Mahler measure and equidistribution

Given a compact set $E \subset \mathbb{C}$, $\mathrm{cap}(E) = 1$, let Ω_E be the unbounded component of $\overline{\mathbb{C}} \setminus E$, and let $g_E(z, \infty)$ be the Green function of Ω_E. For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros, define $m_E(P_n) := \log|a_n| + \sum_{\alpha_k \in \Omega_E} g_E(\alpha_k, \infty)$. Let μ_E be the equilibrium measure of E, and let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$.

Theorem (Bilu, Bombieri, Rumely, Favre and Rivera-Letelier, ...)

If $\lim_{n\to\infty} m_E(P_n)/n = 0$ *then* $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$.

Ex. 1 (Bilu) If $E = D = \{|z| \le 1\}$ then $g_D(z, \infty) = \log|z|$, $m_D(P_n) = \log|a_n| + \sum_{|\alpha_k|>1} \log|\alpha_k|$ and $d\mu_D(e^{i\theta}) = d\theta/(2\pi)$.

Ex. 2 If $E = [0, 4]$ then $g_{[0,4]}(z, \infty) = \log|(z - 2 + \sqrt{z^2 - 4z})/2|$ and $d\mu_{[0,4]}(x) = dx/(\pi\sqrt{x(4-x)}),\ x \in [0, 4]$.

For $P_n(z) = 2T_n(z/2 - 1)$, we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4-x)})$ and

$$\lim_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Mahler measure and equidistribution

Given a compact set $E \subset \mathbb{C}$, $\text{cap}(E) = 1$, let Ω_E be the unbounded component of $\overline{\mathbb{C}} \setminus E$, and let $g_E(z, \infty)$ be the Green function of Ω_E. For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) \in \mathbb{Z}[z]$ with simple zeros, define $m_E(P_n) := \log|a_n| + \sum_{\alpha_k \in \Omega_E} g_E(\alpha_k, \infty)$. Let μ_E be the equilibrium measure of E, and let $\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_k}$.

Theorem (Bilu, Bombieri, Rumely, Favre and Rivera-Letelier, . . .)

If $\lim_{n\to\infty} m_E(P_n)/n = 0$ *then* $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$.

Ex. 1 (Bilu) If $E = D = \{|z| \le 1\}$ then $g_D(z, \infty) = \log|z|$, $m_D(P_n) = \log|a_n| + \sum_{|\alpha_k|>1} \log|\alpha_k|$ and $d\mu_D(e^{i\theta}) = d\theta/(2\pi)$.
Ex. 2 If $E = [0, 4]$ then $g_{[0,4]}(z, \infty) = \log|(z - 2 + \sqrt{z^2 - 4z})/2|$ and $d\mu_{[0,4]}(x) = dx/(\pi\sqrt{x(4 - x)}),\ x \in [0, 4]$.
For $P_n(z) = 2T_n(z/2 - 1)$, we have $\tau_n \xrightarrow{*} dx/(\pi\sqrt{x(4 - x)})$ and

$$\lim_{n\to\infty} \frac{\text{tr}(\alpha)}{n} = \lim_{n\to\infty} \int x\, d\tau_n(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4 - x)}} = 2.$$

Electrostatic centroid and the trace problem

Theorem (Baernstein, Laugesen and Pritsker, 2011)

If $E \subset [0,\infty)$ *is compact,* $\mathrm{cap}(E) \geq 1$*, then*

$$\int x\, d\mu_E(x) \geq \int x\, d\mu_{[0,4]}(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Corollary

If $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$ *then*

$$\liminf_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} \geq \int x\, d\mu_E(x) \geq 2.$$

Remark: This holds for roots of orthogonal and extremal polynomials, backward orbits in monic polynomial dynamics, etc.
Remark: For $\phi(x) : [0,\infty) \to [0,\infty)$ such that $\phi(x^2)$ is convex on $\mathbb{R}$:

$$\int \phi(x)\, d\mu_E(x) \geq \int \phi(x)\, d\mu_{[0,4]}(x)$$

Electrostatic centroid and the trace problem

Theorem (Baernstein, Laugesen and Pritsker, 2011)

If $E \subset [0,\infty)$ is compact, $\text{cap}(E) \geq 1$*, then*

$$\int x\,d\mu_E(x) \geq \int x\,d\mu_{[0,4]}(x) = \int_0^4 \frac{x\,dx}{\pi\sqrt{x(4-x)}} = 2.$$

Corollary

If $\tau_n \xrightarrow{*} \mu_E$ as $n \to \infty$ then

$$\liminf_{n\to\infty} \frac{\text{tr}(\alpha)}{n} \geq \int x\,d\mu_E(x) \geq 2.$$

Remark: This holds for roots of orthogonal and extremal polynomials, backward orbits in monic polynomial dynamics, etc.

Remark: For $\phi(x) : [0,\infty) \to [0,\infty)$ such that $\phi(x^2)$ is convex on $\mathbb{R}$:

$$\int \phi(x)\,d\mu_E(x) \geq \int \phi(x)\,d\mu_{[0,4]}(x)$$

Electrostatic centroid and the trace problem

Theorem (Baernstein, Laugesen and Pritsker, 2011)

If $E \subset [0,\infty)$ *is compact,* $\mathrm{cap}(E) \geq 1$*, then*

$$\int x\,d\mu_E(x) \geq \int x\,d\mu_{[0,4]}(x) = \int_0^4 \frac{x\,dx}{\pi\sqrt{x(4-x)}} = 2.$$

Corollary

If $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$ *then*

$$\liminf_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} \geq \int x\,d\mu_E(x) \geq 2.$$

Remark: This holds for roots of orthogonal and extremal polynomials, backward orbits in monic polynomial dynamics, etc.

Remark: For $\phi(x) : [0,\infty) \to [0,\infty)$ such that $\phi(x^2)$ is convex on $\mathbb{R}$:

$$\int \phi(x)\,d\mu_E(x) \geq \int \phi(x)\,d\mu_{[0,4]}(x)$$

Electrostatic centroid and the trace problem

Theorem (Baernstein, Laugesen and Pritsker, 2011)

If $E \subset [0,\infty)$ *is compact,* $\mathrm{cap}(E) \geq 1$*, then*

$$\int x\, d\mu_E(x) \geq \int x\, d\mu_{[0,4]}(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Corollary

If $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$ *then*

$$\liminf_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} \geq \int x\, d\mu_E(x) \geq 2.$$

Remark: This holds for roots of orthogonal and extremal polynomials, backward orbits in monic polynomial dynamics, etc.
Remark: For $\phi(x) : [0,\infty) \to [0,\infty)$ such that $\phi(x^2)$ is convex on $\mathbb{R}$:

$$\int \phi(x)\, d\mu_E(x) \geq \int \phi(x)\, d\mu_{[0,4]}(x)$$

Electrostatic centroid and the trace problem

Theorem (Baernstein, Laugesen and Pritsker, 2011)

If $E \subset [0, \infty)$ *is compact,* $\operatorname{cap}(E) \geq 1$*, then*

$$\int x\, d\mu_E(x) \geq \int x\, d\mu_{[0,4]}(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Corollary

If $\tau_n \xrightarrow{*} \mu_E$ *as* $n \to \infty$ *then*

$$\liminf_{n\to\infty} \frac{\operatorname{tr}(\alpha)}{n} \geq \int x\, d\mu_E(x) \geq 2.$$

Remark: This holds for roots of orthogonal and extremal polynomials, backward orbits in monic polynomial dynamics, etc.

Remark: For $\phi(x) : [0, \infty) \to [0, \infty)$ such that $\phi(x^2)$ is convex on $\mathbb{R}$:

$$\int \phi(x)\, d\mu_E(x) \geq \int \phi(x)\, d\mu_{[0,4]}(x)$$

Electrostatic centroid and the trace problem

Theorem (Baernstein, Laugesen and Pritsker, 2011)

If $E \subset [0,\infty)$ is compact, $\mathrm{cap}(E) \geq 1$*, then*

$$\int x\, d\mu_E(x) \geq \int x\, d\mu_{[0,4]}(x) = \int_0^4 \frac{x\, dx}{\pi\sqrt{x(4-x)}} = 2.$$

Corollary

If $\tau_n \xrightarrow{*} \mu_E$ as $n \to \infty$ then

$$\liminf_{n\to\infty} \frac{\mathrm{tr}(\alpha)}{n} \geq \int x\, d\mu_E(x) \geq 2.$$

Remark: This holds for roots of orthogonal and extremal polynomials, backward orbits in monic polynomial dynamics, etc.
Remark: For $\phi(x) : [0,\infty) \to [0,\infty)$ such that $\phi(x^2)$ is convex on $\mathbb{R}$:

$$\int \phi(x)\, d\mu_E(x) \geq \int \phi(x)\, d\mu_{[0,4]}(x)$$

Smallest limit points of symmetric means

For $P_n(z) = a_n \prod_{k=1}^{n}(z-\alpha_k) = \sum_{k=0}^{n} a_k z^k \in \mathbb{Z}[z]$, with totaly positive and simple α_k, we consider symmetric means

$$\frac{\sigma_m}{\binom{n}{m}} = \binom{n}{m}^{-1} \sum_{j_1<j_2<\ldots<j_m} \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_m} = (-1)^m \frac{a_{n-m}}{a_n\binom{n}{m}},$$

and define $\ell_m = \liminf_{n\to\infty} \left\{\sigma_m/\binom{n}{m} : \alpha_k > 0 \text{ of degree } n,\ |a_n| \le M\right\}$.

Theorem

If $\tau_n \xrightarrow{*} \mu_E$, where $E \subset [0,\infty)$ with $\mathrm{cap}(E) \ge 1$, then

$$\liminf_{n\to\infty} \frac{\sigma_m}{\binom{n}{m}} \ge \left(\int x\, d\mu_E(x)\right)^m \ge 2^m, \quad m \in \mathbb{N}.$$

Equality holds for the conjugates of $4\cos^2(\pi/(4p))$ (roots of $2T_p(x/2-1)/(x-2)$, where p is an odd prime).
Remark: If $E \subset [0,\infty)$ with $\mathrm{cap}(E) = 1$, then we can allow factors Q with multiplicities of order $o(n)$, provided $m_E(Q) \le C$.

Smallest limit points of symmetric means

For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) = \sum_{k=0}^n a_k z^k \in \mathbb{Z}[z]$, with totaly positive and simple α_k, we consider symmetric means

$$\frac{\sigma_m}{\binom{n}{m}} = \binom{n}{m}^{-1} \sum_{j_1<j_2<\ldots<j_m} \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_m} = (-1)^m \frac{a_{n-m}}{a_n\binom{n}{m}},$$

and define $\ell_m = \liminf_{n\to\infty} \left\{\sigma_m/\binom{n}{m} : \alpha_k > 0 \text{ of degree } n,\ |a_n| \le M\right\}$.

Theorem

If $\tau_n \xrightarrow{*} \mu_E$, *where* $E \subset [0,\infty)$ *with* $\mathrm{cap}(E) \ge 1$, *then*

$$\liminf_{n\to\infty} \frac{\sigma_m}{\binom{n}{m}} \ge \left(\int x\, d\mu_E(x)\right)^m \ge 2^m, \quad m \in \mathbb{N}.$$

Equality holds for the conjugates of $4\cos^2(\pi/(4p))$ (roots of $2T_p(x/2-1)/(x-2)$, where p is an odd prime).

Remark: If $E \subset [0,\infty)$ with $\mathrm{cap}(E) = 1$, then we can allow factors Q with multiplicities of order $o(n)$, provided $m_E(Q) \le C$.

Smallest limit points of symmetric means

For $P_n(z) = a_n \prod_{k=1}^{n}(z-\alpha_k) = \sum_{k=0}^{n} a_k z^k \in \mathbb{Z}[z]$, with totaly positive and simple α_k, we consider symmetric means

$$\frac{\sigma_m}{\binom{n}{m}} = \binom{n}{m}^{-1} \sum_{j_1<j_2<\ldots<j_m} \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_m} = (-1)^m \frac{a_{n-m}}{a_n\binom{n}{m}},$$

and define $\ell_m = \liminf_{n\to\infty} \left\{\sigma_m/\binom{n}{m} : \alpha_k > 0 \text{ of degree } n,\ |a_n| \le M\right\}$.

Theorem

If $\tau_n \xrightarrow{*} \mu_E$, *where* $E \subset [0,\infty)$ *with* $\operatorname{cap}(E) \ge 1$, *then*

$$\liminf_{n\to\infty} \frac{\sigma_m}{\binom{n}{m}} \ge \left(\int x\, d\mu_E(x)\right)^m \ge 2^m, \quad m \in \mathbb{N}.$$

Equality holds for the conjugates of $4\cos^2(\pi/(4p))$ (roots of $2T_p(x/2-1)/(x-2)$, where p is an odd prime).

Remark: If $E \subset [0,\infty)$ with $\operatorname{cap}(E) = 1$, then we can allow factors Q with multiplicities of order $o(n)$, provided $m_E(Q) \le C$.

Smallest limit points of symmetric means

For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) = \sum_{k=0}^n a_k z^k \in \mathbb{Z}[z]$, with totaly positive and simple α_k, we consider symmetric means

$$\frac{\sigma_m}{\binom{n}{m}} = \binom{n}{m}^{-1} \sum_{j_1<j_2<\ldots<j_m} \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_m} = (-1)^m \frac{a_{n-m}}{a_n\binom{n}{m}},$$

and define $\ell_m = \liminf_{n\to\infty} \left\{\sigma_m/\binom{n}{m} : \alpha_k > 0 \text{ of degree } n,\ |a_n| \le M\right\}$.

Theorem

If $\tau_n \xrightarrow{*} \mu_E$, *where* $E \subset [0,\infty)$ *with* $\mathrm{cap}(E) \ge 1$, *then*

$$\liminf_{n\to\infty} \frac{\sigma_m}{\binom{n}{m}} \ge \left(\int x\, d\mu_E(x)\right)^m \ge 2^m, \quad m \in \mathbb{N}.$$

Equality holds for the conjugates of $4\cos^2(\pi/(4p))$ (roots of $2T_p(x/2-1)/(x-2)$, where p is an odd prime).

Remark: If $E \subset [0,\infty)$ with $\mathrm{cap}(E) = 1$, then we can allow factors Q with multiplicities of order $o(n)$, provided $m_E(Q) \le C$.

Smallest limit points of symmetric means

For $P_n(z) = a_n \prod_{k=1}^n (z - \alpha_k) = \sum_{k=0}^n a_k z^k \in \mathbb{Z}[z]$, with totaly positive and simple α_k, we consider symmetric means

$$\frac{\sigma_m}{\binom{n}{m}} = \binom{n}{m}^{-1} \sum_{j_1<j_2<\ldots<j_m} \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_m} = (-1)^m \frac{a_{n-m}}{a_n\binom{n}{m}},$$

and define $\ell_m = \liminf_{n\to\infty} \left\{\sigma_m/\binom{n}{m} : \alpha_k > 0 \text{ of degree } n,\ |a_n| \le M\right\}$.

Theorem

If $\tau_n \xrightarrow{*} \mu_E$, *where* $E \subset [0,\infty)$ *with* $\operatorname{cap}(E) \ge 1$, *then*

$$\liminf_{n\to\infty} \frac{\sigma_m}{\binom{n}{m}} \ge \left(\int x\, d\mu_E(x)\right)^m \ge 2^m, \quad m \in \mathbb{N}.$$

Equality holds for the conjugates of $4\cos^2(\pi/(4p))$ (roots of $2T_p(x/2-1)/(x-2)$, where p is an odd prime).
Remark: If $E \subset [0,\infty)$ with $\operatorname{cap}(E) = 1$, then we can allow factors Q with multiplicities of order $o(n)$, provided $m_E(Q) \le C$.