Galois and Functional Analyses Via Weil Height

Adam M Hughes

Mathematics Department

The University of Texas at Austin

8 October 2015

Outline

Motivation

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

Motivation

The University of Texas at Austin

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

- Distribution of natural orbits of "generic" elements of idèle class groups.

Motivation

TEXAS
The University of Texas at Austin

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

- Distribution of natural orbits of "generic" elements of idèle class groups.

- Recast and generalize classical algebraic theorems in analytic contexts.

Motivation

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

- Distribution of natural orbits of "generic" elements of idèle class groups.

- Recast and generalize classical algebraic theorems in analytic contexts.
    - Measure theory for ideal behaviors

Motivation

TEXAS
The University of Texas at Austin

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

- Distribution of natural orbits of "generic" elements of idèle class groups.

- Recast and generalize classical algebraic theorems in analytic contexts.
  - Measure theory for ideal behaviors
  - Density theorems for Dirichlet's Unit Theorem (Fili, Miner 2014)

Motivation

TEXAS
The University of Texas at Austin

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

- Distribution of natural orbits of "generic" elements of idèle class groups.

- Recast and generalize classical algebraic theorems in analytic contexts.
  - Measure theory for ideal behaviors
  - Density theorems for Dirichlet's Unit Theorem (Fili, Miner 2014)
  - Integration for products over local places

Motivation

TEXAS
The University of Texas at Austin

- Use functional and Fourier analyses on L^1 of compact groups to understand Diophantine approximation on global fields.

- Distribution of natural orbits of "generic" elements of idèle class groups.

- Recast and generalize classical algebraic theorems in analytic contexts.
  - Measure theory for ideal behaviors
  - Density theorems for Dirichlet's Unit Theorem (Fili, Miner 2014)
  - Integration for products over local places

Outline

Notation

- k is a number field.

Notation

- k is a number field.
- $\mathrm{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N},\ \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.

Notation

TEXAS
The University of Texas at Austin

- k is a number field.
- $\text{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N},\ \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.

Notation

- k is a number field.
- $\mathrm{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N},\ \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.
- $\mathbf{x} = (x_v)_v$ denotes an element of I_k.

Notation

- k is a number field.
- $\mathrm{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N}, \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.
- $\mathbf{x} = (x_v)_v$ denotes an element of I_k.

Definition

- $\mathcal{F}$ is the family of finite, Galois extensions of k--partially ordered by inclusion.

Notation

- k is a number field.
- $\text{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N}, \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.
- $\mathbf{x} = (x_v)_v$ denotes an element of I_k.

Definition

- $\mathcal{F}$ is the family of finite, Galois extensions of k--partially ordered by inclusion.
- $I = \varinjlim_{K \in \mathcal{F}} I_K$ is the injective limit of the inductive system of topological groups $\{I_K\}_{K \in \mathcal{F}}$.

Notation

TEXAS
The University of Texas at Austin

- k is a number field.
- $\text{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N},\ \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.
- $\mathbf{x} = (x_v)_v$ denotes an element of I_k.

Definition

- $\mathcal{F}$ is the family of finite, Galois extensions of k--partially ordered by inclusion.
- $I = \varinjlim_{K \in \mathcal{F}} I_K$ is the injective limit of the inductive system of topological groups $\{I_K\}_{K \in \mathcal{F}}$.
- $\mathcal{N}$ is the maximal, compact subgroup of I.

Notation

- k is a number field.
- Tor$(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N}, \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.
- $\mathbf{x} = (x_v)_v$ denotes an element of I_k.

Definition

- $\mathcal{F}$ is the family of finite, Galois extensions of k--partially ordered by inclusion.
- $I = \varinjlim_{K \in \mathcal{F}} I_K$ is the injective limit of the inductive system of topological groups $\{I_K\}_{K \in \mathcal{F}}$.
- $\mathcal{N}$ is the maximal, compact subgroup of I.
- Y is the set of places of $\overline{\mathbb{Q}}$.

Notation

- k is a number field.
- $\mathrm{Tor}(\overline{\mathbb{Q}}^{\times}) = \{\zeta \in \overline{\mathbb{Q}} : \exists n \in \mathbb{N},\ \zeta^n = 1\}$ is the set of torsion points (i.e. roots of unity) of the group $(\overline{\mathbb{Q}}^{\times}, \cdot)$.
- I_k the idèle group associated to k.
- $\mathbf{x} = (x_v)_v$ denotes an element of I_k.

Definition

- $\mathcal{F}$ is the family of finite, Galois extensions of k--partially ordered by inclusion.
- $I = \varinjlim_{K \in \mathcal{F}} I_K$ is the injective limit of the inductive system of topological groups $\{I_K\}_{K \in \mathcal{F}}$.
- $\mathcal{N}$ is the maximal, compact subgroup of I.
- Y is the set of places of $\overline{\mathbb{Q}}$.
- $G_k = \mathrm{Gal}\left(\overline{\mathbb{Q}}/k\right)$ is the absolute Galois group of k with normalized Haar measure dg.

Outline

The Main Theorem

TEXAS
The University of Texas at Austin

Theorem (H.)

(i) $I/\mathcal{N}$ *is a* $\mathbb{Q}$*-vector space with* $\overline{\mathbb{Q}}^\times/Tor(\overline{\mathbb{Q}}^\times)$ *as a* $\mathbb{Q}$*-sub vector space.*

The Main Theorem

Theorem (H.)

(i) $I/\mathcal{N}$ *is a* $\mathbb{Q}$*-vector space with* $\overline{\mathbb{Q}}^{\times}/Tor(\overline{\mathbb{Q}}^{\times})$ *as a* $\mathbb{Q}$*-sub vector space.*

(ii) $I/\mathcal{N}$ *carries a natural, archimedean norm given by the (idèlic) Weil height*

$$h(\mathbf{x}) = \sum_{v} |\log|x_v|_v|.$$

The Main Theorem

TEXAS
The University of Texas at Austin

Theorem (H.)

(i) *$I/\mathcal{N}$ is a $\mathbb{Q}$-vector space with $\overline{\mathbb{Q}}^\times/\mathrm{Tor}(\overline{\mathbb{Q}}^\times)$ as a $\mathbb{Q}$-sub vector space.*

(ii) *$I/\mathcal{N}$ carries a natural, archimedean norm given by the (idèlic) Weil height*

$$h(\mathbf{x}) = \sum_v |\log|x_v|_v|.$$

(iii) *The completion of $I/\mathcal{N}$ with respect to the height norm is isomorphic to $L^1(Y, \mathcal{B}, \lambda)$ with λ a Galois-invariant measure.*

The Main Theorem

TEXAS
The University of Texas at Austin

Theorem (H.)

(i) $I/\mathcal{N}$ *is a* $\mathbb{Q}$*-vector space with* $\overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ *as a* $\mathbb{Q}$*-sub vector space.*

(ii) $I/\mathcal{N}$ *carries a natural, archimedean norm given by the (idèlic) Weil height*

$$h(\mathbf{x}) = \sum_{v} |\log |x_v|_v|.$$

(iii) *The completion of* $I/\mathcal{N}$ *with respect to the height norm is isomorphic to* $L^1(Y, \mathcal{B}, \lambda)$ *with* λ *a Galois-invariant measure.*

(iv) *The completion of* $\overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ *is the co-dimension 1 subspace of* $L^1(Y, \mathcal{B}, \lambda)$ *with mean zero. (Dirichlet's Unit Theorem)*

The Main Theorem

TEXAS
The University of Texas at Austin

Theorem (H.)

(i) $I/\mathcal{N}$ *is a* $\mathbb{Q}$*-vector space with* $\overline{\mathbb{Q}}^{\times}/\text{Tor}(\overline{\mathbb{Q}}^{\times})$ *as a* $\mathbb{Q}$*-sub vector space.*

(ii) $I/\mathcal{N}$ *carries a natural, archimedean norm given by the (idèlic) Weil height*

$$h(\mathbf{x}) = \sum_{v} |\log|x_v|_v|.$$

(iii) *The completion of* $I/\mathcal{N}$ *with respect to the height norm is isomorphic to* $L^1(Y, \mathcal{B}, \lambda)$ *with* λ *a Galois-invariant measure.*

(iv) *The completion of* $\overline{\mathbb{Q}}^{\times}/\text{Tor}(\overline{\mathbb{Q}}^{\times})$ *is the co-dimension 1 subspace of* $L^1(Y, \mathcal{B}, \lambda)$ *with mean zero. (Dirichlet's Unit Theorem)*

(v) $L^1(Y, \mathcal{B}, \lambda)$ *has a natural Banach algebra structure inherited from* $L^1(G_k)$*.*

Annular Decomposition

Definition

- Let v be a place of k. Then we denote by D_v a choice of decomposition group for v with normalized Haar measure dh.

Annular Decomposition

Definition

- Let v be a place of k. Then we denote by D_v a choice of decomposition group for v with normalized Haar measure dh.
- Define the linear operator

$$T_v : L^1(G_k) \to L^1(G_k/D_v), \quad (T_v f)(x) = \int_{D_v} f(xh)\, \mathrm{d}h.$$

Annular Decomposition

TEXAS
The University of Texas at Austin

Definition

- Let v be a place of k. Then we denote by D_v a choice of decomposition group for v with normalized Haar measure dh.
- Define the linear operator

$$T_v : L^1(G_k) \to L^1(G_k/D_v), \quad (T_v f)(x) = \int_{D_v} f(xh)\,\mathrm{d}h.$$

Theorem (H.)

The Banach algebra $L^1(Y, \mathcal{B}, \lambda)$ in Theorem 1 contains the direct image of $\bigoplus_v L^1(G_k)$ under the map $\bigoplus_v T_v$ as a dense subset, i.e.

$$L^1(Y) = \overline{\bigoplus_v T_v\left(L^1(G_k)\right)}.$$

Outline

Dirichlet's Approximation Theorem

TEXAS
The University of Texas at Austin

Definition (Circle Metric)

Let $x \in \mathbb{R}$ and define the function $\|\cdot\| : \mathbb{R} \to \left[0, \frac{1}{2}\right]$ by

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n|$$

Dirichlet's Approximation Theorem

Definition (Circle Metric)

Let $x \in \mathbb{R}$ and define the function $\|\cdot\| : \mathbb{R} \to \left[0, \frac{1}{2}\right]$ by

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n|$$

Theorem (Dirichlet)

Let $x \in \mathbb{R}$. Then

$$\min_{1 \le q \le Q} \|qx\| \le Q^{-1}$$

Idèle class approximation

Theorem (Dirichlet's Unit Theorem)

Let I_K^1 *is the set of idèles of* K *for which* $\prod_w |x_w|_w = 1$*. Then* $K^\times$ *is a co-compact lattice in* I_K^1*.*

Idèle class approximation

TEXAS
The University of Texas at Austin

Theorem (Dirichlet's Unit Theorem)

Let I_K^1 is the set of idèles of K for which $\prod_w |x_w|_w = 1$. Then $K^\times$ is a co-compact lattice in I_K^1.

Question

Can we estimate density in the idèle class group as Dirichlet's theorem does for the reals mod 1?

Idèle class approximation

TEXAS
The University of Texas at Austin

Theorem (Dirichlet's Unit Theorem)

Let I_K^1 is the set of idèles of K for which $\prod_w |x_w|_w = 1$. Then $K^\times$ is a co-compact lattice in I_K^1.

Question

Can we estimate density in the idèle class group as Dirichlet's theorem does for the reals mod 1?

Definition

- Let $\mathcal{N} \cap I_K = \mathcal{N}_K \subseteq I_K^1$ be the set of idèles of K with $|x_w|_w = 1$ for all w.

Idèle class approximation

TEXAS
The University of Texas at Austin

Theorem (Dirichlet's Unit Theorem)

Let I_K^1 is the set of idèles of K for which $\prod_w |x_w|_w = 1$. Then $K^\times$ is a co-compact lattice in I_K^1.

Question

Can we estimate density in the idèle class group as Dirichlet's theorem does for the reals mod 1?

Definition

- Let $\mathcal{N} \cap I_K = \mathcal{N}_K \subseteq I_K^1$ be the set of idèles of K with $|x_w|_w = 1$ for all w.
- S_∞ is the set of infinite places of K, and $|S_\infty| = r + 1$.
- $\varepsilon_1, \ldots, \varepsilon_r$ are multiplicatively independent fundamental units of K.

Idèle class approximation

TEXAS
The University of Texas at Austin

Theorem (Dirichlet's Unit Theorem)

Let I_K^1 is the set of idèles of K for which $\prod_w |x_w|_w = 1$. Then $K^\times$ is a co-compact lattice in I_K^1.

Question

Can we estimate density in the idèle class group as Dirichlet's theorem does for the reals mod 1?

Definition

- Let $\mathcal{N} \cap I_K = \mathcal{N}_K \subseteq I_K^1$ be the set of idèles of K with $|x_w|_w = 1$ for all w.
- S_∞ is the set of infinite places of K, and $|S_\infty| = r + 1$.
- $\varepsilon_1, \ldots, \varepsilon_r$ are multiplicatively independent fundamental units of K.
- $\langle\langle \mathbf{x} \rangle\rangle = \min_{\alpha \in K^\times} h(\mathbf{x}\alpha^{-1}) \longleftarrow$ (analogous to the circle metric on $\mathbb{R}/\mathbb{Z}$)

Radius Bound

Theorem (H.)

Let k be a Galois number field with discriminant D_K. Then with $\mathcal{G}_k = k^\times/\text{Tor}(k^\times)$ and for all $\mathbf{x} \in I_k^1/\mathcal{N}_k$ there is a constant M such that

$$\langle\langle \mathbf{x} \rangle\rangle \leq M.$$

we can provide an explicit bound for the best constant with the bound

$$\langle\langle \mathbf{x} \rangle\rangle \leq \frac{1}{2}\left(\log|D_K| + M'\right).$$

and M' is explict and depends only on the units of K.

Outline

A Basis

Theorem (Krein, Milman, Rutman)

There is a (Schauder) basis for

$$\left\{ f \in L^1(Y, \mathcal{B}, \lambda) : \int_Y f \, d\lambda = 0 \right\}$$

contained in $\overline{\mathbb{Q}}^\times / Tor(\overline{\mathbb{Q}}^\times) \subseteq L^1(Y, \mathcal{B}, \lambda)$.

A Basis

TEXAS
The University of Texas at Austin

Theorem (Krein, Milman, Rutman)

There is a (Schauder) basis for

$$\left\{ f \in L^1(Y, \mathcal{B}, \lambda) : \int_Y f \, d\lambda = 0 \right\}$$

contained in $\overline{\mathbb{Q}}^\times / \mathit{Tor}(\overline{\mathbb{Q}}^\times) \subseteq L^1(Y, \mathcal{B}, \lambda)$.

Application

The implication of the theorem is that there is a (relatively sparse) subset of $\overline{k}^\times$ such that *any* element of $\overline{k}^\times$ can be approximated (in terms of height) to within ϵ of a finite product of elements of this special subset.

Statistics and the CLT

TEXAS
The University of Texas at Austin

Observation (H)

For each $f_{\mathbf{x}} \in I/\mathcal{N}$ and all $1 \leq p \leq \infty$, we have that $f \in L^p(Y, \mathcal{B}, \lambda)$.

Statistics and the CLT

Observation (H)

For each $f_{\mathbf{x}} \in I/\mathcal{N}$ and all $1 \leq p \leq \infty$, we have that $f \in L^p(Y, \mathcal{B}, \lambda)$.

Question

Does some sort of analogue for the Central Limit Theorem hold for some "interesting" subsets of $I/\mathcal{N}$?

Statistics and the CLT

TEXAS
The University of Texas at Austin

Observation (H)

For each $f_{\mathbf{x}} \in I/\mathcal{N}$ and all $1 \leq p \leq \infty$, we have that $f \in L^p(Y, \mathcal{B}, \lambda)$.

Question

Does some sort of analogue for the Central Limit Theorem hold for some "interesting" subsets of $I/\mathcal{N}$?

Application

Compute the variance and other higher moments of assorted algebraic numbers (modulo torsion) embedded in $I/\mathcal{N}$, to see if this statistical information affects approximability or the distribution of the associated numbers in the larger space.

Northcott's Theorem

Theorem (Northcott)

The set of elements of $\overline{\mathbb{Q}}^{\times}$ *of bounded height and degree is finite.*

Northcott's Theorem

TEXAS
The University of Texas at Austin

Theorem (Northcott)

The set of elements of $\overline{\mathbb{Q}}^{\times}$ *of bounded height and degree is finite.*

Question

Can we prove Northcott's theorem using only information furnished to us by the construction of the embedding of $\overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ into $L^1(Y, \mathcal{B}, \lambda)$?

Northcott's Theorem

TEXAS
The University of Texas at Austin

Theorem (Northcott)

The set of elements of $\overline{\mathbb{Q}}^{\times}$ of bounded height and degree is finite.

Question

Can we prove Northcott's theorem using only information furnished to us by the construction of the embedding of $\overline{\mathbb{Q}}^{\times}/\mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$ into $L^1(Y, \mathcal{B}, \lambda)$?

Observation

A field, k, is said to have (B) if there is a constant $c > 0$ such that the set of points of k with height less than c consists solely of torsion points (i.e. roots of unity). A purely topological but **equivalent** topological characterization of fields with (B) is the requirement that $k^{\times}/\mathrm{Tor}(k^{\times})$ be a discrete subset of $L^1(Y, \mathcal{B}, \lambda)$.

Outline

Algebraic relations

TEXAS
The University of Texas at Austin

Observation

The center of $L^1(Y, \mathcal{B}, \lambda)$ contains the functions f_p associated to rational primes. This provides a natural setting for polynomials on the set $\{\log p\}_p$ and ties in to questions of transcendence related to theorems such as Lindemann-Weierstraß, Baker, and the Schanuel conjecture.

Algebraic relations

TEXAS
The University of Texas at Austin

Observation

The center of $L^1(Y, \mathcal{B}, \lambda)$ contains the functions f_p associated to rational primes. This provides a natural setting for polynomials on the set $\{\log p\}_p$ and ties in to questions of transcendence related to theorems such as Lindemann-Weierstraß, Baker, and the Schanuel conjecture.

Application

Fourier analysis on $\mathcal{Z}(L^1(Y))$ could provide a new direction of attack on the algebraic independence of rational prime logarithms (over $\mathbb{Q}$).

Equidistribution

TEXAS
The University of Texas at Austin

Conjecture

Let k be a number field and K/k be finite. Then the $\mathbb{Z}$-orbit of almost every point $\mathbf{x} \in I_K^1/\mathcal{N}_K$ equidistributes modulo $K^\times/\mathrm{Tor}(K^\times)$ relative to the height metric.

Equidistribution

Conjecture

Let k be a number field and K/k be finite. Then the $\mathbb{Z}$-orbit of almost every point $\mathbf{x} \in I_K^1/\mathcal{N}_K$ equidistributes modulo $K^\times/\mathrm{Tor}(K^\times)$ relative to the height metric.

Application

As in the classical case, the dynamics tell us what points are "special," and may be worth extra study. Just as the non-equidistributing point in $\mathbb{T}^n$ are rationals we expect the same to hold in our context due to geometric similarities with an n-torus.

The End

Thank you for listening.