Lifting Problems and Galois Theory (15w5035)

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August 16–21, 2015

1 Objectives

The aim of the BIRS workshop "Lifting Problems and Galois Theory" workshop was to bring together researchers and advanced Ph.D. students working on Galois theory in order to advance this field. The lifting problems for curves which were the focus of the workshop have been a central topic in algebraic geometry since the work of Grothendieck, Deligne and Mumford and others on étale fundamental groups and moduli spaces of curves. The subject has been undergoing rapid change due to the introduction of new techniques, as in the proof of the Oort conjecture.

The workshop had 41 participants. There were 8 one-hour talks, thirteen 45-minute talks, and one 95-minute problem session.

2 One-hour talks

The one-hour talks of the workshop gave introductions to several aspects of lifting problems and Galois theory.

2.1 Frans Oort: Lifting questions.

This talk gave an overview of the main questions and techniques used in the study of lifting problems. The speaker introduced liftings of algebraic curves (with automorphisms), liftings of higher dimensional varieties, and (CM-)liftings of abelian varieties.

2.2 Stefan Wewers: Swan conductors and differential obstructions.

Swan conductors measure ramification of Galois extensions with respect to a valuation. They exist in many different forms. This talk explained how a certain Swan conductor with "differential value" due to Kazuya Kato can be used to define obstructions against lifting Galois covers from characteristic p to characteristic zero. Proving that this obstruction vanishes in certain cases was an important ingredient in the proof of the Oort conjecture on lifting cyclic Galois covers (by Obus, Pop and Wewers). By recent work of Andrew Obus, proving a more general vanishing result is the only obstacle left in proving a "generalized Oort conjecture" on lifting covers with cyclic Sylow p-subgroups.

2.3 David Harbater: Galois group schemes over arithmetic curves.

Much of modern Galois theory takes place in the context of function fields of curves defined over complete discretely valued fields. A common strategy is to choose a projective model of the function field and consider Galois branched covers over the closed fiber, which one then attempts to lift to the whole model. This has been used for the inverse Galois problem, embedding problems, and lifting problems, often with the help of patching methods in order to work locally. Traditionally one considers finite Galois groups (or profinite groups in the limit), but one can also treat non-constant finite Galois group schemes via torsors, as in work of Moret-Bailly. This talk considered more general linear algebraic groups as Galois group schemes over such function fields, in two contexts: the inverse differential Galois problem, and obstructions to local-global principles.

2.4 Robert Guralnick: Groups and curves.

This was a survey talk discussing how finite group theory is useful in studying various problems related to Brauer groups, coverings of curves, automorphism groups of curves and liftings of curves with group actions from positive characteristic to characteristic zero.

2.5 Pierre Dèbes: Specializations of covers and inverse Galois theory.

This talk presented a series of problems and results from a program about the specializations of covers of the line in connection with inverse Galois theory. The main topics included Hilbert's irreducibility theorem, the Inverse Galois Problem and its regular version and the Malle conjecture.

2.6 Kiran Kedlaya: Combinatorial constraints on lifting problems via p-adic differential equations.

This talk described an approach to recovering the standard combinatorial constraints arising in the study of local lifting problems as a corollary of the properties of convergence polygons of p-adic connections.

2.7 Irene Bouw: Computing L-functions of superelliptic curves.

This talk reported on algorithmic results for computing the local L-factor and the conductor exponent of a cyclic cover of the projective line at the primes of bad reduction. As an application, the functional equation was verified numerically for a large class of examples.

2.8 Frans Oort: CM liftings.

This talk explained the full story from Deuring (1941), via Weil, Tate, Honda-Tate, isogenies (1992) and finally results of the recent book (2014) "Complex multiplication and lifting problems" by Ching-Li Chai, Brian Conrad and Frans Oort, giving full answers to possible CM lifting questions. Several proofs were given and complete answers were formulated.

3 45-minute talks

The 45-minute talks of the workshop were given by senior researchers, postdocs and advanced graduate students.

3.1 Lior Bary-Soroker: Geometric versus arithmetic ramification.

Let $f : C \to \mathbb{P}^1$ be a branched covering defined over $\mathbb{Q}$. For $a \in \mathbb{Q}$, the fiber $f^{-1}(a)$ gives rise to a number field (in fact, étale algebra) which, loosely speaking, is generated by the coordinates of the points in the fiber. The main focus of this talk was the study of the number of ramified prime numbers in these number fields. Two results were presented:

(1) a central limit theorem, which answers the question what the typical number of ramification is, and

(2) sharp upper bounds.

The underlying idea behind these results is that the geometric branch locus "controls" the arithmetic one. If time permits, some applications, e.g. to the minimal ramification problem will be discussed.

3.2 Anna Cadoret: Structure of the image of the geometric étale fundamental group on étale cohomology with $\mathbb{F}_\ell$-coefficients.

When studying representations of the arithmetic étale fundamental group on étale cohomology, the knowledge of the structure of the image geometric étale fundamental group plays a crucial part. In particular, when the ring of coefficients is a field, it is conjectured that this image is semi simple. This is known for $\mathbb{Q}_\ell$-coefficients and in characteristic 0. This talk focused on the case of $\mathbb{F}_\ell$-coefficients in positive characteristic.

3.3 Rachel Davis: Galois theory of a quaternion origami.

Let X be equal to an elliptic curve over $\mathbb{Q}$ minus its origin. Let $f : Y \to X$ be an étale cover and let $\overline{x} \in X$ be a geometric point. Grothendieck and others consider Galois representations arising from the action of $G_{\mathbb{Q}}$ on $\left\{f^{-1}(\overline{x})\right\}$. In this talk a particular map f was studied with deck transformation group equal to the quaternion group.

3.4 Armin Holschbach: Étale contractible varieties in positive characteristic.

By Artin-Schreier theory, the affine line $\mathbb{A}^1_k$ over an algebraically closed field k of characteristic $p > 0$ has an infinite fundamental group. This is in contrast to the situation in characteristic 0, where the affine line can be thought of as an algebraic equivalent of the unit interval in topology: Not only is it simply connected, but it is indeed contractible in the sense of étale homotopy theory. Since the affine line as natural candidate does not work, the question arises whether there is any étale contractible variety in positive characteristic. In this talk, it was shown that there are no non-trivial smooth varieties over an algebraically closed field of characteristic p that are étale contractible, and some consequences for the decomposition theory of fundamental groups of varieties in positive characteristic were discussed.

3.5 Aristides Kontogeorgis: Representations of automorphisms and deformation of curves.

In this talk, applications of the representation theory of automorphism groups of curves were given to the theory of deformations of curves with automorphisms.

3.6 Christian Liedtke: Good reduction of $\mathrm{K3}$ surfaces.

By a classical theorem of Serre and Tate, extending previous results of Néron, Ogg, and Shafarevich, an Abelian variety over a p-adic field has good reduction if and only if the Galois action on its first ℓ-adic cohomology is unramified. In this talk, it was shown that if the Galois action on second ℓ-adic cohomology of a K3 surface over a p-adic field is unramified, then the surface admits an "RDP model" over that field, and good reduction (that is, a smooth model) after a finite and unramified extension. (Standing assumption: potential semi-stable reduction for K3's.) Moreover, examples were given where such an unramified extension is really needed. On the way, existence and termination of certain semistable flops were established, and group actions of models of varieties were studied.

3.7 Sophie Marques: Holomorphic differentials for Galois towers of function fields.

In this talk, the necessary conditions were recalled which permitted Boseck to obtain an explicit basis for the space of the holomorphic differentials for Kummer and Artin-Schreier extensions of a rational field. For this, the basics about Kummer and Artin-Schreier extensions were reviewed, particularly the existence of standard forms. Then, it was explained how it is possible to obtain a basis for a Galois tower of function fields of a rational field, provided the existence of a global standard form using Boseck's method. The Galois action on the basis was described and the Galois module structure of the holomorphic differentials for a cyclic function field over a perfect field was presented. This is a natural extension of the results done previously by Sotiris Karanikolopoulos and Aristides Kontogeorgis over an algebraically closed field. Finally, encountered problems for possible further developments/applications were presented.

3.8 Danny Neftin: Monodromy and ramification of rational functions.

The monodromy group and ramification type are two fundamental invariants associated to every rational function. This talk discussed the accumulating work towards describing all possibilities for both the monodromy group and the ramification type of an indecomposable rational function.

3.9 Jennifer Park: Faithful realizability of tropical curves.

Every algebraic curve over a nontrivially valued field has a corresponding tropical curve (through a process called "tropicalization"), where tropical curves are defined as balanced weighted 1-dimensional rational polyhedral complexes. It is then natural to ask whether tropical curves can be realized as the tropicalization of a smooth, complete and connected algebraic curve. Further, the question arises whether the tropicalization can be faithful. In this talk, the basics of the related topics in tropical geometry were first outlined, and then the above question of faithful realizability was answered for a large class of tropical curves.

3.10 Christalin Razafindramahatsiaro: Deuring's constant reductions theory and lifting problems.

Let X be a stable curve over a Dedekind scheme S, with smooth generic fiber X_η. It is well known (from Deligne and Mumford) that there exists a natural injective homomorphism between the full automorphism group of X_η and any special fibre of X. In this talk, first a generalization was given of this theorem in function fields of one variable version. Then, a solution to a "weak lifting problem" for cyclic curves was presented. In particular, the complete list of all full automorphism groups of hyperelliptic curves in odd prime characteristic was given that can be lifted to characteristic 0.

3.11 Zachary Scherr: Separated Belyi Maps.

Let C be a smooth, projective and geometrically irreducible algebraic curve defined over $\mathbb{C}$. In 1980, G. V. Belyi gave an unexpected necessary and sufficient condition for C to be isomorphic to a curve defined over $\overline{\mathbb{Q}}$. Namely, that there should exist a *Belyi map* $\varphi\colon C \to \mathbb{P}^1$. That is, a finite morphism which is ramified only over the three point set $\{0, 1, \infty\}$. This talk was concerned with how much flexibility there is in constructing Belyi maps. For a fixed curve $C/\overline{\mathbb{Q}}$, it is known that for each positive integer n there are, up to automorphism, finitely many n-element subsets of $C(\overline{\mathbb{Q}})$ occurring as the preimage of $\{0, 1, \infty\}$ under a Belyi map. While it is extremely difficult to try to describe all such subsets, this talk discussed a result in this direction. It was proved that given finite, disjoint subsets $S, T \subseteq C(\overline{\mathbb{Q}})$ there is always a Belyi map φ which is ramified on S and for which $\varphi(T) \cap \{0, 1, \infty\} = \emptyset$, refining a theorem of Mochizuki. This talk discussed both this theorem and a comparable theorem in positive characteristic.

3.12 Jeroen Sijsling: On descent of marked curves and maps.

Let F be a field with separable closure F^{sep}, and let X be a curve over F^{sep} that is isomorphic with all its $\mathrm{Gal}(F^{\mathrm{sep}} \,|\, F)$-conjugates. Then one can wonder whether there exists a descent of X, that is, a curve X_0 over F that is isomorphic with X over F. Surprisingly, counterexamples due to Shimura and Earle show that such a descent need not always exist. However, classical results by Dèbes and Emsalem imply that the statement does hold for smoothly marked curves. More precisely, let X be a curve as above and let $P \in X(F^{\mathrm{sep}})$ be a smooth point on X. Then if for all $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}} \,|\, F)$ there exists an isomorphism $\sigma X \to X$ taking σP to P, then there exists a curve X_0 over F and a point $P_0 \in X_0(F)$ such that (X_0, P_0) is isomorphic to (X, P)

over F^{sep}. In this talk, a constructive version of this classical result was discussed that uses the branches of a morphism between algebraic curves. This allows to remove some superfluous hypotheses and to give explicit descent constructions for marked curves and Belyĭ maps. After showing these examples and their applications, some counterexamples were given for singular curves.

3.13 Michael Zieve: Monodromy groups in Galois theory.

This talk presented several types of results obtained by means of monodromy groups. These include refinements of Hilbert's irreducibility theorem, results about images of morphisms of curves over finite fields, results about reducibility of fibered products, and solutions to certain functional equations in rational or meromorphic functions.

4 Open Problems

In the 95-minute problem session, the following 13 open problems were discussed.

4.1 Weak/almost local Oort (David Harbater)

This focuses on the local lifting problem.

Question 1 (Weak local Oort): Which groups G have the property that some covers lift?

Question 2 (Almost local Oort): Which groups G have the property that all covers with sufficiently large conductor lift?

For a variant which seems more tractable, replace "sufficiently large conductor" by "sufficiently large first jump". Note that the groups G in question are always of the form $G \cong P \rtimes \mathbb{Z}/m\mathbb{Z}$, where P is a p-group and $p \nmid m$.

Known results:

(i) $p > 2$, $G = (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 2$: This is known to be a weak local Oort group. There is a necessary condition in order to lift, given by certain congruence conditions on the ramification.

(ii) $p = 2$, $G = Q_8$ or G a generalized quaternion group: This might be an almost local Oort group. However, it is not a full local Oort group, and moreover it is not even known whether it is weak. Note that in this case the KGB obstruction to being almost local Oort vanishes.

(iii) If G contains a subgroup of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ with $m > 1$ and $p \nmid m$, then G is not weak local Oort.

4.2 Weyl CM curves (Frans Oort)

Let L be a field with $[L : \mathbb{Q}] = 2g$. Then L is called a Weyl CM field if it is a CM field and its Galois closure $\tilde{L}$ has the maximal possible degree for a CM field, namely $(2g)(g!)$. If so,

$$\mathrm{Gal}(\tilde{L}/\mathbb{Q}) \cong W_g := (\mathbb{Z}/2)^g \ltimes \mathrm{Sym}_g.$$

An algebraic curve over $\mathbb{C}$ is called a Weyl CM curve if its Jacobian is a Weyl CM abelian variety.

Question: Does there exist a Weyl CM algebraic curve for any $g \geq 4$? Can we give examples of such curves?

Remark 1. The automorphism group of a Weyl CM curve is trivial in case the curve is non-hyperelliptic; if a Weyl CM curve is hyperelliptic, the automorphism group is generated by the hyperelliptic involution. We know that we can construct CM curves by choosing curves with "many automorphisms". We see that this is of no help for the case of Weyl CM curves.

Remark 2.

(i) Coleman conjectured that for a fixed $g \geq 4$ the number of CM curves of genus g is finite [3, Conjecture 6].

(ii) For $4 \leq g \leq 7$, however, this is not correct, as was shown for $4 \leq g \leq 6$ by de Jong and Noot [6]; for a complete survey see [9].

(iii) Chai and Oort showed a modified version of Colemans conjecture: for a fixed $g \geq 4$ the number of Weyl CM curves of genus g is finite (conditional under the André-Oort condition); see [2, Proposition 3.7]. Also see [17].

4.3 Mumford curves in the Torelli locus (Frans Oort)

Let $\mathcal{A}_4$ be the moduli space of principally polarized abelian varieties over $\mathbb{C}$. Mumford constructed Shimura varieties (special subvarieties) of dimension 1 in $\mathcal{A}_4$, see [10]. Any geometric generic point corresponds to an abelian variety with endomorphism ring $\mathbb{Z}$. Hecke correspondences can be applied to obtain countably many of such curves. The (closure of the) image $\mathcal{T}_g$ of the Torelli morphism $\mathcal{M}_g \to \mathcal{A}_g$ is called the (closed) Torelli locus.

Question: Is any of these curves contained in the Torelli locus $\mathcal{T}_4 \subset \mathcal{A}_4$?

More generally, one can ask for Shimura varieties in $\mathcal{A}_g$ contained in $\mathcal{T}_g$, meeting the open Torelli locus $\mathcal{T}_g^0 \subset \mathcal{T}_g$. For a discussion, and some references see [9].

4.4 Good, ordinary reduction (Frans Oort / Folklore)

Question: Let A be an abelian variety over a number field K. Does there exist a prime p of K where A has good ordinary reduction?

In case $\dim(A) \leq 2$ the answer is affirmative, as proved on [12, pp. 370-372]; also see [18]. For a (potentially) CM abelian variety the Newton polygons of the reductions can be determined, and in this case primes of good, ordinary reduction do exist.

4.5 Irreducible polynomials with unit coefficients (Lior Bary-Soroker)

Let S_n be the set of polynomials of the form

$$f(x) = x^n \pm x^{n-1} \pm \cdots \pm 1 \tag{1}$$

that are irreducible.

Question: What proportion of polynomials of the form (1) are irreducible? In other words, what is the value of

$$\lim_{n\to\infty} \frac{\#S_n}{2^n} \ ?$$

The limit is conjectured to be 1. Known results:

(i) Poonen considered polynomials with coefficients in $\{0, 1\}$ and showed that the liminf is at least $1/n$.

(ii) Konyagin improved this bound to a (non-zero) multiple $1/\log(n)$.

We can also ask these questions modulo large p. For $p = 2$, the generalized Riemann hypothesis gives results for infinitely many n. There is also some numerical evidence. (Note that this question was also posed by "some guy on the street" [sic] on MathOverflow.)

4.6 Wild one-point covers of the projective line (David Harbater)

Fix $g \in \mathbb{Z}_{\geq 0}$, and let k be an algebraically closed field of characteristic $p > 0$.

Question 1: For which finite groups G is there a G-Galois branched cover $f : Y \to \mathbb{P}^1$ over k with Y of genus g and with f étale outside ∞?

Questions 2 and 3 below are variants of Question 1.

Question 2: For which finite groups G is there a cover $f : Y \to \mathbb{P}^1$ over k with Y of genus g that is étale outside ∞ and such that the monodromy group of f is isomorphic to G?

Question 3: Fix a quasi-p group G. What is the smallest positive integer g for which there exists a cover $f : Y \to \mathbb{P}^1$ over k with Y of genus g that is étale outside ∞ and such that the monodromy group of f is isomorphic to G?

4.7 Existence of connected torsors for finite group schemes (Ted Chinburg)

Question: Does there exist a finite group scheme G over $\mathbb{Q}$ such that there does not exist a connected G-torsor over $\mathbb{Q}$?

Remark. This reduces to the inverse Galois problem theory when considering constant group schemes.

For a variant, one can replace $\mathbb{Q}$ by any number field.

4.8 Lifting an automorphism to a normal domain (Frans Oort)

Question: Do there exist a field $\kappa \supset \mathbb{F}_p$, a complete, non-singular, absolutely irreducible algebraic curve C over κ, and an automorphism $b \in \mathrm{Aut}_k(C)$ such that (C, b) does not lift to any mixed characteristic normal local domain with residue field κ?

Is the question different for a perfect κ, or for a finite κ?

Remark. Let $\kappa \supset \mathbb{F}_p$ be an algebraically closed field, D a complete, non-singular, irreducible algebraic curve over κ, and let $b \in \mathrm{Aut}_\kappa(D)$. By Oort-Sekiguchi-Suwa, Green-Matignon, Obus-Wewers [11] and Pop [16] we know that (D, b) can be lifted to characteristic zero. Hence in the situation above (C, b) can be lifted to a (complete, local) mixed characteristic $R \to \kappa$ with residue class field κ. The question asks whether this can be done with a normal ring R.

Brian Conrad pointed out that if a lifting is possible to a mixed characteristic domain, this need not imply lifting is possible to a normal domain. Note that normalization of an integral domain may extend a residue class field.

4.9 Uniform boundedness of rational preimages (Michael Zieve)

Question: Fix positive integers d and D, and geometrically irreducible d-dimensional varieties X_1, X_2 defined over a degree-D number field K. Does there exist an integer N such that, for each finite morphism $\phi : X_1 \to X_2$ defined over K, the induced map on rational points $\phi : X_1(K) \to X_2(K)$ is at most N-to-1 over all points outside a proper Zariski-closed subset of $X_2(\overline{K})$? Further, can N be chosen to depend only on d and D, and not on X_1, X_2 or K?

Remark. The uniform boundedness conjecture for rational torsion on abelian varieties is equivalent to the special case of this question in which ϕ varies over all multiplication-by-n endomorphisms on abelian varieties.

Known results:

(i) Such an integer N exists (and depends only on D) when X_1 and X_2 are genus-1 curves [8].

(ii) Such an integer N exists when $X_1 = X_2 = \mathbb{A}^1$ [1]. In this case the smallest value N can take is the largest m for which K contains the real part of the m-th cyclotomic field; in particular, if $D = 1$ and $X_1 = X_2 = \mathbb{A}^1$ then the optimal value is $N = 6$.

(iii) In case $X_1 = X_2 = \mathbb{P}^1$, if $\deg(\phi)$ is sufficiently large compared to D and also ϕ admits no nontrivial factorization through an intermediate curve, then we can take N to be 2. In fact we could take N to be 1 if we exclude maps $\phi = \psi \circ x^a(x-1)^b \circ \mu$ where $\deg(\mu) = 1$ and $\gcd(a, b) = 1$, see [4].

Remark. It would be interesting to study the analogous question when K is a function field, for instance when $d = 1$. One difficulty is ruling out the possibility that the fibered product of several copies of X_1 (fibered over X_2) could have isotrivial components not contained in the fat diagonal, even though ϕ itself is not isotrivial.

4.10 Fields of definition of endomorphism rings (Kiran Kedlaya)

Question: Let A be an abelian variety of dimension g over a number field K. Let $L|K$ be the minimal extension over which all endomorphisms of A are defined. (This extension is finite Galois.)

Question: If we fix g and vary A (and possibly K), then what is the optimal bound $N(g)$ for $[L : K]$?

The current guess is that this should be $2^g g!$ for $g >> 0$.

Known results:

(i) $N(1) = 2$;

(ii) $N(2) = 48$;

(iii) $N(g) \leq 2(9n)^{2g}$ (by work of Silverberg).

4.11 Complete subvarieties of moduli spaces (Frans Oort)

Question 1: Fix $g \geq 4$. Which are the complete subvarieties $W \subset \mathcal{A}_g \otimes \mathbb{F}_p$ of codimension g?

Remark.

(i) We study complete subvarieties of $W \subset \mathcal{A}_g \otimes k$ for some algebraically closed field k. It is known that the codimension of W is at least g, as was proved by van der Geer, see [5, Theorem 3.3]. This means $\dim(W) \leq g(g-1)/2$ for a complete subvariety.

(ii) Infinitely many codimension g subvarieties exist for $g = 0$ and $g = 1$ in any characteristic.

(iii) The locus $V_0(\mathcal{A}_g \otimes \mathbb{F}_p)$ of abelian varieties of dimension g with p-rank zero is of codimension g and it is complete, see [13, Theorem 1.1].

(iv) In [14, Question 14B], we find the conjecture that $\mathcal{A}_g \otimes \mathbb{C}$ does not contain a complete subvariety of codimension g for any $g \geq 3$.

(v) This conjecture was proved to be true by Keel and Sadun, [7].

(vi) In [15, 14.2], infinitely many complete subvarieties of dimension 3 inside $\mathcal{A}_g \otimes \mathbb{F}_p$ were constructed. However we do not see how to perform an analogous construction for $g > 3$ and obtain infinitely many complete subvarieties of codimension g.

Question 2: Fix $g \geq 4$. Is it true that $V_0(\mathcal{A}_g \otimes \mathbb{F}_p)$ is the only complete subvariety $W \subset \mathcal{A}_g \otimes \mathbb{F}_p$ of codimension g?

Remark. Suppose this question has an affirmative answer for a given $g \geq 4$ and infinitely many values of p. Then for this value of g it follows that $\mathcal{A}_g \otimes \mathbb{C}$ does not contain a complete subvariety of codimension g (and in this way reproving the Keel-Sadun result).

4.12 The p-rank 0 strata of the Torelli locus (Rachel Pries)

For a prime number p and natural number g, consider the moduli space $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$ of principally polarized abelian varieties of dimension g in characteristic p and consider the Torelli locus $\mathcal{T}_g \subset \mathcal{A}_g$. Let $\mathcal{A}_g^0$ be the subscheme that parametrizes abelian varieties of dimension g with p-rank 0. For $g \geq 3$, it is known that $\mathcal{A}_g^0$ is irreducible, that its generic point has a-number 1, and that its Newton polygon has slopes $1/g$ and $(g-1)/g$. Let $\mathcal{T}_g^0 = \mathcal{T}_g \cap \mathcal{A}_g^0$.

Question 1: For p prime and $g \geq 3$, is $\mathcal{T}_g^0$ irreducible? For the generic point of each of its components, is it true that the a-number is 1 and the Newton polygon has slopes $1/g$ and $(g-1)/g$?

The answer is yes when $g = 3$ and some information is known when $g = 4$.

4.13 Characterizing lifted covers (David Harbater)

Question: Fix a prime p. Which Galois branched covers of $\mathbb{P}^1_{\mathbb{C}}$ whose ramification indices are all prime to p are lifts of smooth Galois branched covers of $\mathbb{P}^1$ in characteristic p?

For a variant, we can fix the ramification type and ask whether there exists a location of branch points that gives a positive answer above.

Known result: If the Galois group is of order prime to p, then all such covers are lifts. Beyond this case the Question is wide open.

Remark. This question essentially concerns to the study of π_1^{tame} in characteristic p. A closely related conjecture is as follows:

Conjecture (Ihara, Kyoto 2010): Let $W_p = W(\overline{\mathbb{F}_p})$ be the ring of Witt vectors over $\overline{\mathbb{F}_p}$, and let $K_p = Q(W_p)$ be the fraction field of W_p, that is, the completion of the maximal unramified extension of $\mathbb{Q}_p$. Let $f : X \to \mathbb{P}^1_{K_p}$ be a G-Galois cover with branch locus $\{0, 1, \infty\}$. If there is a K_p-point of X whose fiber is totally split, then f has potentially good reduction.

This conjecture is true for G solvable, and we can reduce it to the case where G is simple.

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