MULTIVARIATE OPERATOR THEORY
BANFF 2015

Introduction

The workshop *Multivariate Operator Theory* gathered at Banff, in April 2015, experts in function theory of several complex variables, operator algebras, operator theory, free probability, non-commutative analysis and representation theory of Lie groups. The proportion of young researchers among all participants was unusually high, bringing from them the enthusiasm and fresh ideas so much needed in a mature field.

The main themes of the workshop are well illustrated by the summaries below. They contain competitive new results, covering a large array of subjects and opening quite a few directions for future research. The vibrant atmosphere of the workshop laid the basis for new collaborations and interdisciplinary exchanges.

Kenneth Davidson, University of Waterloo
Ronald G. Douglas, Texas A & M Universsity
Jörg Eschmeier, University of Saarland
John William Helton, University of California at San Diego
Mihai Putinar, University of California at Santa Barbara, Newcastle University
(organizers)

Noncommutative Reproducing Kernel Hilbert Spaces

Joseph A. Ball, Virginia Tech
joint with Gregory Marx,Virginia Tech, and Victor Vinnikov, Ben Gurion U.

A prominent theme at this meeting has been the "free analysis" or the study of freely noncommutative functions. As laid out in the book of Kaliuzhnyi-Verbovetskyi–Vinnikov [KVV], one can formulate this notion in the following general setting. We let $\mathcal{V}$ be a vector space and let $\mathcal{V}_{\rm nc} = \amalg_{n=0}^{\infty} \mathcal{V}^{n \times n}$ be the disjoint union over $n \in \mathbb{N}$ of $n \times n$ matrices over $\mathcal{V}$. A subset Ω of a $\mathcal{V}$ is said to be a **noncommutative set** (or **nc set** for short) if $Z, W \in \Omega \Rightarrow \left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right] \in \Omega$. A function from Ω to the set $\mathcal{V}_{0,\rm nc}$ (where $\mathcal{V}_0$ is a second vector space) is said to be a **noncommutative (nc) function** if the following conditions hold:

(1) f is **graded**, i.e., $f \colon \Omega_n \to \mathcal{V}_{0,n} = (\mathcal{V}_0)^{n \times n}$,
(2) f **respects direct sums**, i.e.,

$$Z, W \in \Omega \Rightarrow f\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right]\right) = \begin{bmatrix} f(Z) & 0 \\ 0 & f(W) \end{bmatrix},$$

and
(3) f **respects similarities**, i.e.: whenever $Z, \widetilde{Z} \in \Omega_n$, $A \in \mathbb{C}^{n \times n}$ with A invertible such that $\widetilde{Z} = AZA^{-1}$, then

$$f(\widetilde{Z}) = Af(Z)A^{-1}.$$

An early result in the book [KVV] is that it then follows that, for $Z, W \in \Omega$ and $\left[\begin{smallmatrix} Z & X \\ 0 & W \end{smallmatrix}\right] \in \Omega$, then $f\left(\left[\begin{smallmatrix} Z & X \\ 0 & W \end{smallmatrix}\right]\right)$ has the form

$$f\left(\left[\begin{smallmatrix} Z & X \\ 0 & W \end{smallmatrix}\right]\right) = \begin{bmatrix} f(Z) & \Delta_R f(Z,W)(X) \\ 0 & f(W) \end{bmatrix}$$

where $\Delta_R f(Z,W)$ is linear in the argument X. Such functions enjoy some additional properties, versions of which we shall see presently in the definition of noncommutative kernel coming next. These **right difference-differential operators** $\Delta_R f$ are examples of an object of independent interest called a *first-order noncommutative function.* Suffice to say that in case the set of points Ω is equipped with an adjoint operator $Z \mapsto Z^*$ and if K is a noncommutative kernel as defined in the next paragraph, then $\widetilde{K}$ given by $\widetilde{K}(Z,W)(P) = K(Z,W^*)(P)$ is a nc function of the first order.

We now suppose that $\mathcal{V}_0$ and $\mathcal{V}_1$ are operator spaces—in practice they often appear as C^*-algebras—and that K is a function form $\Omega \times \Omega$ into $\mathcal{L}(\mathcal{V}_1, \mathcal{V}_0)$ (bounded linear operators from $\mathcal{V}_1$ into $\mathcal{V}_0$). We say that K is a **nc kernel** if K has the following properties:

(1) K is **graded**,

$$Z \in \Omega_n,\, W \in \Omega_m \Rightarrow K(Z,W) \in \mathcal{L}(\mathcal{V}_1^{n \times m}, \mathcal{V}_0^{n \times m}),$$

(2) K **respects direct sums**: for $Z \in \Omega_n$, $\widetilde{Z} \in \Omega_{\widetilde{n}}$, $W \in \Omega_m$, $W \in \Omega_{\widetilde{m}}$, $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right] \in \mathcal{A}^{(n+m)\times(\widetilde{n}+\widetilde{m})}$, we have

$$K\left(\left[\begin{smallmatrix} Z & 0 \\ 0 & \widetilde{Z} \end{smallmatrix}\right], \left[\begin{smallmatrix} W & 0 \\ 0 & \widetilde{W} \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]\right) = \begin{bmatrix} K(Z,W)(P_{11}) & K(Z,\widetilde{W})(P_{12}) \\ K(\widetilde{Z},W)(P_{21}) & K(\widetilde{Z},\widetilde{W})(P_{22}) \end{bmatrix}. \qquad (1)$$

(3) K **respects similarities**:

$$Z, \widetilde{Z} \in \Omega_n,\ A \in \mathbb{C}^{n\times n} \text{ invertible with } \widetilde{Z} = AZA^{-1},$$
$$W, \widetilde{W} \in \Omega_m, B \in \mathbb{C}^{m\times m} \text{ invertible with } \widetilde{W} = BWB^{-1},$$
$$P \in \mathcal{A}^{n\times m} \Rightarrow K(\widetilde{Z}, \widetilde{W})(P) = A\, K(Z,W)(A^{-1}PB^{-1*})\, B^*.$$

We denote the class of all such nc kernels by $\widetilde{\mathcal{T}}^1(\Omega; \mathcal{V}_{0,\mathrm{nc}}, \mathcal{V}_{1,\mathrm{nc}})$. In addition we say that a nc kernel $K \in \widetilde{\mathcal{T}}^1(\Omega; \mathcal{V}_0, \mathcal{V}_1)$ is **completely positive** (cp) if in addition, for all $n \in \mathbb{N}$ we have

$$Z \in \Omega_n,\ P \succeq 0 \text{ in } \mathcal{V}_1^{n\times n} \Rightarrow K(Z,Z)(P) \succeq 0 \text{ in } \mathcal{V}_0^{n\times n}. \tag{2}$$

Completely positive kernels incorporate as special cases variety of more classical notions: (1) completely positive maps between operator systems or C^*-algebras, (2) positive kernels in the sense of Aronszajn, (3) a previous synthesis of the first two notions: a completely positive kernel in the sense of Barreto-Bhat-Liebscher-Skeide. Our main result is a characterization of cp nc kernels paralleling the well-known classical theory. For this result we take $\mathcal{V}_0$ to be the C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{Y}$ and we take $\mathcal{V}_1$ to be a C^*-algebra $\mathcal{A}$.

Theorem 1. *Then the following are equivalent.*

(1) *K is a cp nc kernel from $\Omega \times \Omega$ to $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\mathrm{nc}}$.*

(2) *There is a Hilbert space $\mathcal{H}(K)$ whose elements are nc functions $f\colon \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{Y})_{\mathrm{nc}}$ such that*

$$\langle f(W)(v^*), y\rangle_{\mathcal{Y}^m} = \langle f, K_{W,v,y}\rangle_{\mathcal{H}(K)} \tag{3}$$

where $W \in \Omega_m$, $v \in \mathcal{A}^{1\times m}$, $y \in \mathcal{Y}^m$ and where $K_{W,v,y}\colon \Omega_n \to \mathcal{L}(\mathcal{A}, \mathcal{Y})^{n\times n} \cong \mathcal{L}(\mathcal{A}^n, \mathcal{Y}^n)$ is defined by

$$K_{W,v,y}(Z)u = K(Z,W)(uv)y \tag{4}$$

for $Z \in \Omega_n$, $u \in \mathcal{A}^n$. Furthermore $\mathcal{H}(K)$ is equipped with a unital $*$-representation σ mapping $\mathcal{A}$ to $\mathcal{L}(\mathcal{H}(K))$ given by

$$(\sigma(a)f)\,(W)(v^*) = f(W)(v^*a) \tag{5}$$

(3) *K has a Kolmogorov decomposition: there is a Hilbert space $\mathcal{X}$ equipped with a unital $*$-representation $\sigma\colon \mathcal{A} \to \mathcal{L}(\mathcal{X})$ together with a nc function $H\colon \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{Y})_{\mathrm{nc}}$ so that*

$$K(Z,W)(P) = H(Z)(\mathrm{id}_{\mathbb{C}^{n\times m}} \otimes \sigma)(P)H(W)^* \tag{6}$$

for all $Z \in \Omega_n$, $W \in \Omega_m$, $P \in \mathcal{A}^{n\times m}$.

We note the equivalence of (1) and (3) can be seen as a generalization of the Stinespring dilation theorem.

One can go on to talk about multipliers on nc reproducing kernel Hilbert spaces, associated de Branges-Rovnyak kernels, interpolation problems for contractive multipliers and the notion of complete Pick kernel in this free noncommutative setting.

[KVV] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of Noncommutative Function Theory*, Mathematical Surveys and Monographs **199**, Amer. Math. Soc., Providence, 2014.

Algebraic pairs of commuting isometries

Hari Bercovici[1], Indiana University

Abstract. Pairs of commuting isometries that satisfy a polynomial equation were studied by Agler, Knese and McCarthy [1] under the assumption that both isometries are pure. It was shown by Timko [7] that the purity assumption essentially follows from the polynomial equation. The method involves a model for commuting isometries which was developed by Douglas, Foias, and the author [2–4]. We briefly sketch this model and Timko's argument.

Suppose that V_1 and V_2 are two isometric operators acting on the Hilbert space $\mathcal{H}$ such that $V_1V_2 = V_2V_1$. Given a polynomal with complex coefficients $p \in \mathbb{C}[z_1, z_2]$, the operator $p(V_1, V_2)$ is obtain by substituting V_1 and V_2 for the variables z_1 and z_2. The pair $V = (V_1, V_2)$ is said to be *algebraic* if $p(V_1, V_2) = 0$ for some nonzero $p \in \mathbb{C}[z_1, z_2]$. When V_1 and V_2 are unitary operators, the condition $p(V_1, V_2) = 0$ is satisfied precisely when the joint spectrum of (V_1, V_2) is contained in the zero set of p. When V_1 is a pure isometry and V_2 is unitary, the condition $p(V_1, V_2) = 0$ is easily seen to imply that V_2 has finite spectrum, V_2 itself satisfies an algebraic equation. We therefore dismiss these two special cases which are not particularly interesting. In general (see [4] as well as [5, 6]) the space $\mathcal{H}$ has an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(4)}$$

into reducing spaces for the pair V so that the restrictions $V_1^{(j)} = V_1|\mathcal{H}^{(j)}$ and $V_2^{(j)} = V_2|\mathcal{H}^{(j)}$ have the following properties:

(0) there are no nonzero reducing subspaces $\mathcal{M}$ for the pair $(V_1^{(0)}, V_2^{(0)})$ such that either $V_1^{(0)}|\mathcal{M}$ is unitary or $V_2^{(0)}|\mathcal{M}$ is unitary;
(1) $V_1^{(1)}$ is unitary and $V_2^{(1)}$ is pure;
(2) $V_1^{(2)}$ is pure and $V_2^{(2)}$ is unitary; and
(3) $V_1^{(3)}$ and $V_2^{(3)}$ are both unitary.

The algebraic pairs of interest satisfy $\mathcal{H}^{(j)} = \{0\}$ for $j = 1, 2, 3$, so $\mathcal{H} = \mathcal{H}^{(0)}$ for such pairs. Isometric pairs with $\mathcal{H} = \mathcal{H}^{(0)}$ are constructed as follows. We start with a Hilbert space $\mathcal{F}$, a pure isometry S acting on $\mathcal{F}$, and an operator A of norm at most 1 commuting with S. Denote by U the minimal unitary extension of S, and let B be the unique operator commuting with U and extending A. We denote by $\mathcal{D}$ the defect space of B, that is, the closure of the range of $I - B^*B$, and set

$$\mathcal{H}^{(S,U)} = \mathcal{F} \oplus \mathcal{D} \oplus \mathcal{D} \oplus \cdots ,$$

where the dots represent the orthogonal sum of countably many copies of $\mathcal{D}$. Now we define commuting isometries $V_1^{(S,U)}$ and $V_2^{(S,U)}$ acting on $\mathcal{H}^{(S,U)}$ as follows:

$$\begin{aligned} V_1^{(S,U)}(f \oplus d_0 \oplus d_1 \oplus \cdots) &= Sf \oplus Ud_0 \oplus Ud_1 \oplus \cdots , \\ V_2^{(S,U)}(f \oplus d_0 \oplus d_1 \oplus \cdots) &= Af \oplus (I - A^*A)^{1/2} f \oplus d_0 \oplus d_1 \oplus \cdots , \end{aligned}$$

for every vector $f \oplus d_0 \oplus d_1 \oplus \cdots \in \mathcal{H}^{(S,U)}$.

Every pair $V = (V_1, V_2)$ for which $\mathcal{H} = \mathcal{H}^{(0)}$ can be unitarily identified with a pair of the form $(V_1^{(S,A)}, V_2^{(S,A)})$ with the additional property that the pair (S, A)

[1]The author was supported in part by a grant from the National Science Foundation.

has no reducing subspace $\mathcal{M}$ such that $A|\mathcal{M}$ is unitary. The relevant observation for algebraic isometries is this: if A is not already isometric (in which case $\mathcal{D} = \{0\}$) then the pair $(V_1^{(S,A)}, V_2^{(S,A)})$ has an invariant subspace (namely, $\{0\} \oplus \mathcal{D} \oplus \mathcal{D} \oplus \cdots$) on which $V_1^{(S,A)}$ is an *absolutely continuous* unitary. As noted above, this implies that the pair $(V_1^{(S,A)}, V_2^{(S,A)})$ is not algebraic. We conclude that an algebraic pair V of commuting isometries satisfying $\mathcal{H} = \mathcal{H}^{(0)}$ must be such that V_1 is pure and, by symmetry, V_2 is pure as well. This is the essence of Timko's argument [7].

Algebraic pairs of commuting pure isometries were shown in [1] to satisfy a special kind of polynomial equation. A plausible model theory is also suggested in [1], at least in the case of finite cyclic multiplicity. These questions, as well as their analogs for more than two commuting isometries, remain open. The model theory developed in [4] allows one to construct general n-tuples of commuting isometries for arbitrary integers n. These models may well prove to be useful in the study of algebraic n-tuples.

References

[1] L. Agler, G. Knese, and J. E. McCarthy, Algebraic pairs of isometries. *J. Operator Theory* **67** (2012), 215–236.

[2] H. Bercovici, R. G. Douglas, and C. Foias, On the classification of multi-isometries. *Acta Sci. Math.* (*Szeged*) **72** (2006), 639–661.

[3] ———, Bi-isometries and commutant lifting, *Characteristic functions, scattering functions and transfer functions*, 51–76, Oper. Theory Adv. Appl., **197**, Birkhäuser Verlag, Basel, 2010.

[4] ———, Canonical models for bi-isometries, *A panorama of modern operator theory and related topics*, 177–205, Oper. Theory Adv. Appl., **218**, Birkhäuser/Springer Basel AG, Basel, 2012.

[5] R. E. Curto and F.-H. Vasilescu, Standard operator models in the polydisc. *Indiana Univ. Math. J.* **42** (1993), 791–810.

[6] ———-, Standard operator models in the polydisc. II, *Indiana Univ. Math. J.* **44** (1995), 727–746.

[7] E. Timko, Private communication.

Operators on Matrix Weighted L^2 Spaces

Kelly Bickel[2], Bucknell University

The main topic of interest is the behavior of operators on $L^2(W)$, where W is a matrix weight. To set the scene, we first consider the scalar situation.

Scalar Setting. We say w is a *weight* if $w \in L^1_{loc}(\mathbb{R})$ and $w(x) > 0$ almost everywhere. Then $L^2(w) \equiv \left\{ f : \int |f|^2 w < \infty \right\}$. Moreover, a weight w is an A_2 *weight* if

$$[w]_{A_2} \equiv \sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-1} \right) \equiv \sup_I \langle w \rangle_I \left\langle w^{-1} \right\rangle_I < \infty.$$

The operators we consider are Calderón-Zygmund operators (CZO's) and for our purposes, the most important example is the Hilbert transform. Now, it is well-known that a CZO T is bounded on $L^2(w)$ if w is an A_2 weight. Moreover, this is an if and only if condition if T is sufficiently nice, for example if T is the Hilbert transform. This led mathematicians to study the dependence of the operator norm

[2]Research supported in part by National Science Foundation grant DMS # 1448846

$\|T\|_{L^2(w)\to L^2(w)}$ on $[w]_{A_2}$. After many intermediate studies, Hytönen proved: if T is a CZO and w is an A_2 weight, then $\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$.

One of the main tools used by Hytönen is the reduction of general CZO's to dyadic shifts. To define these, we require the dyadic grid $\mathcal{D}$, the Haar basis $\{h_I\}_{I\in\mathcal{D}}$, and the distance on $\mathcal{D}$ obtained by connecting each interval I to its children I_+ and I_- to form a tree and setting $d_{tree}(I,J)$ = minimum number of edges between I and J on the tree.

Then, a dyadic shift of complexity r is a basically a CZO that sends each Haar function h_I to a weighted combinations of other Haar functions h_J satisfying $d_{tree}(I,J) \leq r$. An extremely useful fact is that all CZO's can be written as "averages" of dyadic shifts. That makes the following result due to Lacey-Petermichl-Reguera particularly nice: If T is a dyadic shift, then $\|T\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$. They proved this by applying a variation of the following $T(1)$ Theorem, which is due to Nazarov-Treil-Volberg, to dyadic shifts:

Theorem 1. *Let w, v be weights and let T be a band operator of radius r. If for all $I \in \mathcal{D}$*

$$\|TM_w \mathbf{1}_I\|_{L^2(v)} \leq A\|\mathbf{1}_I\|_{L^2(w)} \text{ and } \|T^*M_v \mathbf{1}_I\|_{L^2(w)} \leq A\|\mathbf{1}_I\|_{L^2(v)},$$

then the following bound holds:

$$\|T\|_{L^2(w^{-1})\to L^2(v)} \lesssim 2^{2r}A.$$

Here, T a band operator of radius r means that $T \in \mathcal{B}(L^2(\mathbb{R}))$ and

$$\langle Th_I, h_J\rangle_{L^2} = 0 \quad \text{whenever } d_{\text{tree}}(I,J) > r.$$

We are interested in these type of results in the matrix setting.

Matrix Setting. We say W is a *matrix weight* if W is a $d \times d$ matrix-valued function with entries in $L^1_{loc}(\mathbb{R})$ and $W(x) > 0$ almost everywhere. Then $L^2(W) \equiv \{f : \mathbb{R} \to \mathbb{C}^d : \int \|W^{\frac{1}{2}} f\|^2 < \infty\}$. A $d \times d$ matrix weight W is an A_2 *weight* if

$$[W]_{A_2} \equiv \sup_I \left\| \langle W\rangle_I^{\frac{1}{2}} \left\langle W^{-1}\right\rangle_I^{\frac{1}{2}} \right\|^2 < \infty.$$

The operators of interest are matrix CZO's on $L^2(\mathbb{R}, \mathbb{C}^d)$, which are just scalar CZO's applied to each component of the vector-valued functions separately.

It is already known that many scalar results generalize to this setting. For example, Treil-Volberg showed that the Hilbert Transform H is bounded on $L^2(W)$ if and only if $W \in A_2$. Similarly, Nazarov-Treil and Volberg showed that a CZO T is bounded on $L^2(W)$ if $W \in A_2$. However, this leaves the question: What is the dependence of $\|T\|_{L^2(W)\to L^2(W)}$ on $[W]_{A_2}$?

The answer is currently unknown, but some related bounds have been established. For the Hilbert transform, we obtained this result, joint with B. Wick and S. Petermichl [1]:

$$\|H\|_{L^2(W)\to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log[W]_{A_2}.$$

For $\mathcal{S}$ a sparse operator, we obtained the following estimate with B. Wick [3]:

$$\|\mathcal{S}\|_{L^2(W)\to L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}}.$$

Here a sparse operator S is of the form $Sf \equiv \sum_{I\in\mathfrak{S}} \langle f\rangle_I \mathbf{1}_1$, where $\mathfrak{S} \subseteq \mathcal{D}$ and for each $I \in \mathfrak{S}$, $\sum_{J\in ch_{\mathfrak{S}}(I)} |J| \leq \frac{1}{2}|I|$, where the sum is over the maximal elements of $\mathfrak{S}$ strictly contained in I. Sparse operators are important because as shown by Conde-Alonso and Rey, Lerner, and Lacey, scalar CZO's can be controlled by sparse operators.

One other main goal was to generalize the previously discussed $T(1)$ Theorem to the matrix setting. Fo similar band operators, joint with B. Wick [2], we established:

Theorem 2. *Let W, V be A_2 weights and let T be a band operator with radius r. If for all $I \in \mathcal{D}$ and $e \in \mathbb{C}^d$,*

$$\|TM_W \mathbf{1}_I e\|_{L^2(V)} \leq A\|\mathbf{1}_I e\|_{L^2(W)} \text{ and } \|T^* M_V \mathbf{1}_I e\|_{L^2(W)} \leq A\|\mathbf{1}_I e\|_{L^2(V)},$$

then the following bound holds:

$$\|T\|_{L^2(W^{-1})\to L^2(V)} \lesssim 2^{2r} A\left([W]_{A_2} + [V]_{A_2}\right).$$

This theorem requires $W, V \in A_2$ and depends on $[W]_{A_2}$ and $[V]_{A_2}$. However, that dependence could be removed if one had an optimal Matrix Carleson Embedding Theorem. This and the previous discussion motivate the following questions:

1. Is there a Matrix Carleson Embedding Theorem that does not require $W \in A_2$ and/or does not depend on $[W]_{A_2}$?
2. Can one establish the $T(1)$ testing conditions for dyadic shifts in the matrix setting to prove a version of the Lacey-Petermichl-Reguera result for matrix dyadic shifts?
3. Can one estimate matrix CZO's using sparse operators, as in the scalar setting?

References

[1] K. Bickel, S. Petermichl, and B. D. Wick. Bounds for the Hilbert Transform with Matrix A_2 Weights. *preprint*, available at *http://arxiv.org/abs/1402.3886*.

[2] K. Bickel and B. D. Wick. Well-Localized Operators on Matrix Weighted L^2 Spaces. *preprint*, available at *http://arxiv.org/abs/1407.3819*.

[3] K. Bickel and B. D. Wick. A Study of the Matrix Carleson Embedding Theorem with Applications to Sparse Operators. *preprint*, available at *http://arxiv.org/abs/1503.06493*.

Duality and peak interpolation for multipliers of the Drury-Arveson space

Raphaël Clouâtre, U. Waterloo

Joint work with Kenneth R. Davidson

1. Henkin measures and the dual space of the ball algebra

Let $A(\mathbb{B}_d)$ denote the ball algebra. A measure μ on the sphere $\mathbb{S}_d$ is $A(\mathbb{B}_d)$–*Henkin* if $\lim_{n\to\infty} \int_{\mathbb{S}_d} f_n d\mu = 0$ whenever $\{f_n\}_n \subset A(\mathbb{B}_d)$ is a bounded sequence converging to 0 in the weak-$*$ topology of $H^\infty(\mathbb{B}_d)$. Every representing measure for some $\lambda \in \mathbb{B}_d$ is an $A(\mathbb{B}_d)$–Henkin measure.

From the operator theoretic point of view, $A(\mathbb{B}_d)$–Henkin measures are interesting because of the following connection. Let $T \in B(\mathcal{H})$ be a contraction. Then, the polynomial functional calculus $p \mapsto p(T)$ can be extended to a contractive homomorphism $\Phi_T : H^\infty(\mathbb{D}) \to B(\mathcal{H})$ which is weak-$*$ continuous if and only if the spectral measure of the unitary part of T is an $A(\mathbb{D})$–Henkin measure.

These measures are classical objects and are fairly well understood. Indeed, a theorem of Henkin, Cole and Range says that a measure is $A(\mathbb{B}_d)$–Henkin if and only it is absolutely continuous with respect to some positive representing measure for the origin. By combining this characterization with the decompositions of measures due to Valskii and Glicksberg-König-Seever, we obtain that

$$A(\mathbb{B}_d)^* = H^\infty(\mathbb{B}_d)_* \oplus_1 TS(\mathbb{S}_d)$$

where $TS(\mathbb{S}_d)$ is the space of measures on $\mathbb{S}_d$ which are singular with respect to every positive representing measure for the origin. Such measures are said to be $A(\mathbb{B}_d)$*-totally singular.*

2. Application to multivariate operator theory

Let $T_1, \ldots, T_d \in B(\mathcal{H})$ be commuting operators such that $T = (T_1, \ldots, T_d)$ is a row contraction. The basic question motivating our work is the following: when can we extend the polynomial functional calculus $p \mapsto p(T)$ to a contractive homomorphism $\Phi_T : \mathcal{M}_d \to B(\mathcal{H})$ which is weak-$*$ continuous? Here $\mathcal{M}_d$ is the multiplier algebra of the Drury-Arveson space H^2_d. Let $\mathcal{A}_d$ denote the closure of the polynomials in $\mathcal{M}_d$.

By analogy with the solution for a single operator, we want to characterize the functionals $\Psi \in \mathcal{A}_d^*$ with the property that $\lim_{n\to\infty} \Psi(f_n) = 0$ whenever $\{f_n\}_n \subset \mathcal{A}_d$ converges to 0 in the weak-$*$ topology of $\mathcal{M}_d$. Such functionals are said to be $\mathcal{A}_d$*–Henkin*. The following is from [2] and should be compared with the corresponding result for $A(\mathbb{B}_d)$ mentioned in the previous section.

Theorem 1. *There is a commutative von Neumann algebra* $\mathfrak{W}$ *such that*

$$\mathcal{A}_d^{**} \cong \mathcal{M}_d \oplus \mathfrak{W}.$$

In particular

$$\mathcal{A}_d^{*} \cong \mathcal{M}_{d*} \oplus_1 \mathfrak{W}_*.$$

Moreover, given $\Psi \in \mathfrak{W}_*$ *there exists an* $A(\mathbb{B}_d)$*–totally singular measure* μ *with* $\|\mu\| = \|\Psi\|$ *such that*

$$\Psi(f) = \int_{\mathbb{S}_d} f d\mu, \quad f \in \mathcal{A}_d.$$

In accordance with the last statement, we say that a measure μ is $\mathcal{A}_d$*–totally singular* if the associated integration functional Ψ lies in $\mathfrak{W}_*$.

Conjecture 1.

- *A measure* μ *is* $A(\mathbb{B}_d)$*–totally singular if and only if it is* $\mathcal{A}_d$*–totally singular.*
- *A measure* μ *is* $A(\mathbb{B}_d)$*–Henkin if and only if it is* $\mathcal{A}_d$*–Henkin.*

These two conjectures are actually equivalent as shown in [2].

3. Peak interpolation

Let X be a compact Hausdorff space and let $A \subset C(X)$ be a closed subspace. A closed subset $K \subset X$ is called a *peak interpolation set for* A if for every $h \in C(K)$ there exists $f \in A$ such that

- $f = h$ on K
- $|f(x)| < \|h\|_K$ for every $x \in X \setminus K$.

In the case of the ball algebra $A(\mathbb{B}_d)$, a classical theorem of Carleson-Rudin-Bishop asserts that any closed subset $K \subset \mathbb{S}_d$ with the property that $|\mu|(K) = 0$ for every $A(\mathbb{B}_d)$–Henkin measure μ must be a peak interpolation set.

We are interested in describing the closed subsets of the sphere where peak interpolation can be achieved using multipliers from $\mathcal{A}_d$. This is a rather subtle question, partly because the obvious inclusion $\mathcal{A}_d \to C(\mathbb{S}_d)$ does not have closed range. Nevertheless, by carefully examining the extreme points of $\mathcal{A}_d^*$ and using Choquet type integral representations for functionals, we can show the following (see [2]).

Theorem 2. *Let $K \subset \mathbb{S}_d$ be a closed subset with the property that $|\mu|(K) = 0$ for every $\mathcal{A}_d$–Henkin measure μ. Then, for every $h \in C(K)$ and every $\epsilon > 0$ there exists $f \in \mathcal{A}_d$ such that*

- *$f = h$ on K*
- *$|f(\zeta)| < \|h\|_K$ for every $\zeta \in \mathbb{S}_d \setminus K$*
- *$\|f\|_{\mathcal{A}_d} \leq (1 + \epsilon)\|h\|_K$.*

We remark that the novel and crucial point is the third property: we can control the multiplier norm of f in terms of the supremum norm of h on the set K. Moreover, assuming that the aforementioned conjecture holds, this result immediately gives a significant improvement of the classical Carleson-Rudin-Bishop theorem.

Finally, let us mention that the assumption of the theorem is certainly satisfied whenever K is closed and countable. Note that achieving peak interpolation in $\mathcal{A}_d$ on such a set is far from being trivial and we do not know how to do it directly.

References

[1] R. Clouâtre and K. Davidson, *Duality, convexity and peak interpolation in the Drury-Arveson space*, preprint, arXiv:1504.00665

Analytic continuation of Toeplitz operators

Miroslav Engliš, Prague & Opava
joint work with H. Bommier and E.-H. Youssfi, Marseille

The standard weighted Bergman spaces on the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$, $n \geq 1$, are defined by

$$A^2_\alpha(\mathbf{B}^n) = \{f \in L^2(\mathbf{B}^n, d\mu_\alpha) : f \text{ holomorphic on } \mathbf{B}^n\},$$
$$d\mu_\alpha(z) := \frac{\Gamma(\alpha+n+1)}{\pi^n\Gamma(\alpha+1)}(1-|z|^2)^\alpha \, dz, \qquad \alpha > -1,$$

where dz denotes the Lebesgue measure on $\mathbf{C}^n$, the constant factor is chosen so as to make $d\mu_\alpha$ a probability measure (so that $\|\mathbf{1}\|_\alpha = 1$), and the restriction $\alpha > -1$ guarantees that the spaces do not reduce just to constant zero.

These spaces possess the reproducing kernel

$$K_\alpha(x, y) = (1 - \langle x, y\rangle)^{-\alpha-n-1}.$$

The norm of a holomorphic function $f(z) = \sum_\nu f_\nu z^\nu$ in A^2_α can be expressed in terms of Taylor coefficients via

$$\|f\|^2_\alpha := \sum_{\nu \text{ multiindex}} |f_\nu|^2 \frac{\nu!\Gamma(\alpha + n + 1)}{\Gamma(|\nu| + \alpha + n + 1)},$$

and $f \in A^2_\alpha$ iff $\|f\|_\alpha < \infty$.

Now the right-hand side actually makes sense and is a positive definite kernel not only for $\alpha > -1$, but even for all $\alpha > -n-1$; denoting the corresponding norm still by $\|f\|_\alpha$ and defining

$$A^2_\alpha := \{f \text{ holomorphic on } \mathbf{B}^n : \|f\|_\alpha < \infty\},$$

we thus obtain an "analytic continuation" of the weighted Bergman spaces A^2_α from $\alpha > -1$ to $\alpha > -n - 1$.

These spaces will still be reproducing kernel Hilbert spaces (RKHS) of holomorphic functions on $\mathbf{B}^n$, with reproducing kernels given by the same formula as before. In particular, we obtain in this way as special cases the Hardy space $H^2(\partial\mathbf{B}^n, d\sigma)$ for $\alpha = -1$, and the Drury-Arveson space, well-known in multivariable operator theory, for $\alpha = -n$.

This kind of "analytic continuation" of weighted Bergman spaces is well known also in the more general setting of *bounded symmetric domains*, where Rossi and Vergne in 1970's showed that A^2_α can be extended to α in the so-called *Wallach set*

$$W_\Omega := (s, \infty) \cup \{j\tfrac{a}{2} - p, j = 0, 1, \dots, r - 1\},$$

where r, p and a are the *rank*, the *genus* and the *characteristic multiplicity* of the domain, respectively, and $s = -\frac{n}{r} - 1 = \frac{r-1}{2}a - p$.

Returning to the ball, recall that a Toeplitz operator $T^{(\alpha)}_\phi$ with symbol $\phi \in L^\infty(\mathbf{B}^n)$ is the operator on $A^2_\alpha(\mathbf{B}^n)$, $\alpha > -1$, defined by

$$T^{(\alpha)}_\phi f = P_\alpha(\phi f),$$

where $P_\alpha : L^2(\mathbf{B}^n, d\mu_\alpha) \to A^2_\alpha$ is the orthogonal projection. For $\alpha = -1$, when $A^2_{-1} = H^2(\partial\mathbf{B}^n, d\sigma)$ is the Hardy space, there is an analogous definition of Toeplitz operators with symbols $\phi \in L^\infty(\partial\mathbf{B}^n, d\sigma)$ rather than $\phi \in L^\infty(\mathbf{B}^n)$, and P_α replaced by the projection in $L^2(\partial\mathbf{B}^n, d\sigma)$ onto $H^2(\partial\mathbf{B}^n, d\sigma)$.

This talk addresses the question of extending, or "analytically continuing", the Toeplitz operators $T^{(\alpha)}_\phi$ also to the spaces A^2_α with $\alpha < -1$. Since in that case the inner product in A^2_α no longer comes from an ambient L^2 space, there is no P_α and $T^{(\alpha)}_\phi f = P_\alpha(\phi f)$ does not make sense. However, it may still happen that $T^{(\alpha)}_\phi f$, for f in some dense subset (e.g. for f a polynomial), is given by an expression which depends holomorphically on α and extends by analyticity to the range $\alpha > -n-1$; so one then has a (densely defined) operator $T^{(\alpha)}_\phi$ even for these α, which in the sense just described is an "analytic continuation" of the original Toeplitz operator.

One can extend the range of α even to all $\alpha \in \mathbf{R}$ by noticing that

$$A^2_\alpha(\mathbf{B}^n) = W^{-\alpha/2}_{\text{hol}}(\mathbf{B}^n)$$

coincindes with the holomorphic Sobolev space of order $-\frac{\alpha}{2}$; likewise, one can then ask about the "analytic continuation" of Toeplitz operators to arbitrary real α.

Our main results are the following. Let $\Omega \subset \mathbf{C}^n$ be a strictly pseudoconvex domain with smooth ($= C^\infty$) boundary, and ρ a positively-signed defining function for Ω. For $\alpha > -1$, the space

$$A^2_{\alpha,\rho} := \{f \in L^2(\Omega, c_{\alpha,\rho}\rho^\alpha \, dz) : f \text{ holomorphic on } \Omega\}$$

where $c_{\alpha,\rho} := \left(\int_\Omega \rho^\alpha \, dz\right)^{-1}$, is nontrivial and is also a RKHS, with reproducing kernel denoted by $K_{\alpha,\rho}(x, y)$. It was shown by the speaker in 2010 that there exists a set $U_\rho \subset \mathbf{C}$ without accumulation point such that $K_{\alpha,\rho}(x, y)$ extends to a holomorphic function of $\alpha \in \mathbf{C} \setminus U_\rho$, for all $x, y \in \Omega$. This extension will still be denoted $K_{\alpha,\rho}(x, y)$.

Theorem 1. *$K_{\alpha,\rho}(x, y)$ actually continues to be a positive definite kernel also for some $\alpha < -1$; that is, the set $W_{\Omega,\rho}$ of all such real α contains the interval $(-1-\epsilon, \infty)$ for some $\epsilon > 0$.*

For $\alpha \in W_{\Omega,\rho}$, we can thus define in the standard manner $A^2_{\alpha,\rho}$ as the RKHS on Ω with reproducing kernel $K_{\alpha,\rho}(x, y)$.

Theorem 2. *$A^2_{\alpha,\rho} = W^{-\alpha/2}_{hol}(\Omega)$, $\forall \alpha \in W_{\Omega,\rho}$.*

We can therefore define

$$A^2_{\alpha,\#} := W^{-\alpha/2}_{\text{hol}}(\Omega), \qquad \forall \alpha \in \mathbf{R},$$

with $A^2_{\alpha,\#} = A^2_{\alpha,\rho}$ for $\alpha > -1$ (as sets, with equivalent norms).

Theorem 3. *$T^{(\alpha)}_\phi$ exists and is bounded on $A^2_{\alpha,\#}$ if $\phi \in C^k(\Omega)$ has bounded derivatives of order not exceeding k, or at least bounded derivatives in the normal directions, and $\alpha > -k-1$.*

The main ingredient in the proofs is the machinery, developed by Boutet de Monvel and Guillemin, of Hardy-space Toeplitz operators with pseudodifferential symbols, combined with properties of the Poisson extension operator.

On the essential normality of principal submodules of the Drury-Arveson module

Quanlei Fang, CUNY
joint work with Jingbo Xia

Let $\mathbf{B}$ be the open unit ball in $\mathbf{C}^n$, $n \geq 2$. Recall that the Drury-Arveson space H^2_n is naturally a Hilbert module over the polynomial ring $\mathbf{C}[z_1, \ldots, z_n]$. A decade ago, Arveson raised the question of whether graded submodules $\mathcal{M}$ of H^2_n are essentially normal, which is now called the Arveson conjecture. That is, for the restricted operators $Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}, \quad 1 \leq j \leq n$, on $\mathcal{M}$, do commutators $[Z^*_{\mathcal{M},j}, Z_{\mathcal{M},i}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > n$? Later in [6], Douglas proposed analogous, but more refined essential normality problems for submodules of the Bergman module $L^2_a(\mathbf{B}, dv)$. Ever since, these essential normality problems have become a very active area of research interest ([1-11]).

In a breakthrough [5], Douglas and Wang showed that for every $q \in \mathbf{C}[z_1, \dots, z_n]$, the submodule $[q]$ of the Bergman module $L^2_a(\mathbf{B}, dv)$ is p-essentially normal for $p > n$. What is remarkable about this result is that it is *unconditional* in the respect that it makes no assumptions about the polynomial q. This led to our earlier work [8], where we showed that the analogous essential normality also holds for every polynomial-generated submodule $[q]$ of the Hardy module $H^2(S)$.

Continuing our earlier investigation [8], the emphasis here is on submodules of the Drury-Arveson module H^2_n. In the case of two complex variables, we show that for every polynomial $q \in \mathbf{C}[z_1, z_2]$, the submodule $[q]$ of H^2_2 is p-essentially normal for $p > 2$. In the case of three complex variables, we show that there is a significant class of $q \in \mathbf{C}[z_1, z_2, z_3]$ for which the submodule $[q]$ of H^2_3 is p-essentially for $p > 3$.

The difficulties involved in the proofs of these results are determined by the weight t $(-n \leq t < \infty)$ of the space involved. Our earlier work [8] covered the range $-2 < t < \infty$, which was enough to settle the problem for all polynomial-generated submodules of the Hardy module $H^2(S)$. Here we first solve the problem unconditionally for the weight range $-3 < t \leq -2$, a consequence of which is the H^2_2-result mentioned above. We then consider the weight $t = -3$, which requires a substantial amount of additional work. At the moment we are only able to solve the $t = -3$ problem under a technical restriction on q, giving us the partial H^2_3-result mentioned above.

References

[1] W. Arveson, The Dirac operator of a commuting d-tuple, J. Funct. Anal. **189** (2002), 53-79.

[2] W. Arveson, p-summable commutators in dimension d, J. Operator Theory **54** (2005), 101-117.

[3] K. Davidson, The mathematical legacy of William Arveson, J. Operator Theory **68** (2012), 307-334.

[4] R. Douglas, X. Tang and G. Yu, An analytic Grothendieck Riemann Roch theorem, preprint 2014, arXiv:1404.4396

[5] R. Douglas and K. Wang, Essential normality of cyclic submodule generated by any polynomial, J. Funct. Anal. **261** (2011), 3155-3180.

[6] R. Douglas, A new kind of index theorem, Analysis, geometry and topology of elliptic operators, 369-382, World Sci. Publ., Hackensack, NJ, 2006.

[7] M. Engliš and J. Eschmeier, Geometric Arveson-Douglas conjecture, Adv. Math. **274** (2015), 606-630.

[8] Q. Fang and J. Xia, Essential normality of polynomial-generated submodules: Hardy space and beyond, J. Funct. Anal. **265** (2013), 2991-3008.

[9] Q. Fang and J. Xia, On the essential normality of principal submodules of the Drury-Arveson module, preprint 2015.

[10] M. Kennedy, Essential normality and the decomposability of homogeneous submodules, Trans. Amer. Math. Soc. 367 (2015), 293-311.

[11] O. Shalit, Stable polynomial division and essential normality of graded Hilbert modules, J. Lond. Math. Soc. (2) **83** (2011), 273-289.

Thin interpolating sequences

Pamela Gorkin, Bucknell University

1. Function theory

R.C. Buck proposed studying the existence of interpolating sequences. He conjectured that if a sequence of points in the unit disk, $\mathbb{D}$, approached the boundary quickly enough, it would be interpolating for the algebra H^∞ of bounded analytic functions on $\mathbb{D}$; that is, for all $(w_n) \in \ell^\infty$ there exists $f \in H^\infty$ such that $f(z_n) = w_n$ for all n. We briefly discuss the background on interpolating sequences before turning to thin interpolating sequences.

In 1958, Carleson [4] presented a condition for a sequence to be interpolating for H^∞; if (z_n) is a Blaschke sequence and B the corresponding Blaschke product, then (z_n) is interpolating if there exists $\delta > 0$ such that $\inf_n(1 - |z_n|^2)|B'(z_n)| \geq \delta > 0$. In 1961, Shapiro and Shields [17] considered interpolation in the Hardy space H^2 and described it as a weighted interpolation problem. A great deal of work followed, including J. P. Earl's [8] description of functions that do the interpolation as well as an estimate on the H^∞-norm of the function, the construction of Per Beurling functions in uniform algebras, and the explicit construction of such Beurling functions by Peter Jones [12]. Earl's norm estimate depends on the constant δ in the definition of interpolation and is close to 1 if δ is close to 1.

But H^∞ is a uniform algebra and its maximal ideal space, $M(H^\infty)$, or the space of nonzero multiplicative linear functionals with the weak-$*$ topology, played a prominent role in its study. By studying certain partitions, mathematicians were able to shed light on the behavior of functions in H^∞. The disk, $\mathbb{D}$, can be identified with a subset of the maximal ideal space, but $M(H^\infty)\setminus\mathbb{D}$ is difficult to understand. As luck would have it and as Sarason showed, the space $H^\infty + C$, consisting of sums of (boundary) functions in H^∞ and continuous functions on $\partial\mathbb{D}$ is a closed algebra [16] with maximal ideal space precisely $M(H^\infty) \setminus \mathbb{D}$. In this set, the Gleason parts play a prominent role: For $m_1, m_2 \in M$ the pseudo-hyperbolic distance is

$$\rho(m_1, m_2) = \sup\{|\hat{f}(m_2)| : f \in H^\infty, \|f\|_\infty \leq 1, \hat{f}(m_1) = 0\}.$$

Points are in the same Gleason part if $\rho(m_1, m_2) < 1$ and this defines an equivalence relation on $M(H^\infty)$. One equivalence class is the unit disk and the others lie in $M(H^\infty + C)$. In trying to understand the parts in $M(H^\infty) \setminus \mathbb{D}$, Hoffman showed that if we consider an interpolating sequence with the property that $\lim_n(1 - |z_n|^2)|B'(z_n)| = 1$, then any part in the closure of this sequence will be homeomorphic to the unit disk. Returning to Buck's original point, if the sequence tends to the boundary very quickly, we should get much more than an interpolating sequence. These sequences are now called *thin sequences*. An equivalent definition of thin is useful: writing B_j for the subproduct of B with the j-th zero deleted and $\delta_j = |B_j(z_j)|$ the thin condition asks that $\delta_j \to 1$.

Thin sequences have zeros that are pseudohyperbolically far apart in the disk as well as in $M(H^\infty + C)$. They are also *indestructible*; that is, if you take an automorphism $\varphi_a(z) = (z - a)/(1 - \overline{a}z)$ (with $a \in \mathbb{D}$) and consider $\varphi_a \circ B$, this will again be a thin Blaschke product (though finitely many zeros may be repeated). But more is true.

Early on in this study, R. Douglas recognized that the beautiful behavior of Sarason's algebra $H^\infty + C$ might be the rule rather than an exception. He asked whether every closed subalgebra B of L^∞ containing H^∞ was generated by H^∞ and the conjugates of inner functions. The answer to Douglas's question was even better than that: Chang and Marshall [6,14] showed that every such algebra was generated by H^∞ and the conjugates of the interpolating Blaschke products invertible in that algebra.

Sticking with the uniform algebra point of view for a moment, one might wonder what happens when one looks at the closed algebra A of H^∞ and the conjugates of all thin interpolating Blaschke products. Hedenmalm [11] showed that an inner function is invertible in A if and only if it is a finite product of thin interpolating Blaschke products.

Thin interpolating sequences turn out to be very well behaved: Wolff and Sundberg [20], [18] showed, among other things, that these sequences are the interpolating sequence for the (very small) algebra $QA = \overline{H^\infty + C} \cap H^\infty$ (here the bar denotes the complex conjugate).

What about an analogue of the Shapiro and Shields result for thin sequences? In [9] the authors showed that an interpolating sequence is a thin sequence if and only if given $(w_n) \in \ell^\infty$, $\|w\|_\infty = 1$ there is a function $f \in H^\infty$ with $\|f\| = 1$ and $|f(z_n) - w_n| \to 0$. Equivalent definitions of thin sequences in the H^p setting were presented in [10]. Dyakonov and Nicolau, [7], showed that an interpolating sequence is thin if and only if there is a sequence (m_j), $0 < m_j < 1$ and $m_j \to 1$ such that every interpolation problem $F(z_j) = w_j$ with $|w_j| \leq m_j$ has a solution $f \in H^\infty$ with $\|F\| \leq 1$. In fact, this happens if and only if there exists a sequence of positive numbers (ε_j) such that every interpolation problem with $1 \geq |a_j| \geq \varepsilon_j$ for all j has a nonvanishing solution $g \in H^\infty$.

2. Operator theory

For $\varphi \in L^\infty$ define the Toeplitz operator on H^2 by $T_\varphi f = P\varphi f$, where P is the orthogonal projection from L^2 to H^2. The Hankel operator is $H_\varphi f = (I-P)\varphi f$, $f \in H^2$. In 1963, Brown and Halmos [3] showed that if $f, g \in L^\infty$, then $T_f T_g = T_{fg}$ if and only if $\overline{f} \in H^\infty$ or $g \in H^\infty$. A natural question is the following: For which symbols f, g is $T_f T_g$ a compact perturbation of a Toeplitz operator? In [2], Axler, Chang and Sarason showed that if $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C$, then $H_{\overline{f}}^\star H_g$ is compact. Though they proved necessity for a large class of functions, the theorem was completed in 1982 by A. Volberg [19]. These proofs relied on the maximal ideal space structure.

Let $k_z(w) = 1/(1-\overline{z}w)$ denote the reproducing kernel for H^2, g_z the normalized reproducing kernel, and let G denote the Gram matrix with entries $g_{ij} = \langle g_i, g_j \rangle$. Volberg showed (see also [5]) that a sequence is thin if and only if the Gram matrix $G = I + K$ with K compact and I the identity. That is, $\delta_j \to 1$ if and only if $G - I$ is compact. The proof used maximal ideal space techniques as well as Hoffman's theory. Volberg also showed that $G - I \in S_2$ where S_2 denotes the Hilbert Schmidt operators if and only if $\prod_j \delta_j$ converges. Thus, $G - I$ is in the Schatten class S_2 if and only if $\sum_j (1 - \delta_j) < \infty$. What about $2 < p < \infty$? Using Earl's theorem and results that are essentially in Shapiro and Shields (see also [1]) J. E. McCarthy, S. Pott, B. Wick and the author showed that for $2 \leq p < \infty$, $G - I \in S_p$ if and only if $\sum_n (1 - \delta_n^2)^{p/2} < \infty$, extending Volberg's theorem to the cases between 2 and infinity as well as simplifying the proof for the case $p = \infty$. Related work for truncated Toeplitz operators appears in [13].

References

[1] J. Agler and J.E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, American Mathematical Society, Providence, 2002.

[2] Sheldon Axler, Sun-Yung A. Chang, and Donald Sarason, *Products of Toeplitz operators*, Integral Equations Operator Theory **1** (1978), no. 3, 285–309.

[3] Arlen and Halmos Brown P. R., *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

[4] L. Carleson, *An interpolation problem for bounded analytic functions*, American J. Math. **80** (1958), 921–930.

[5] I. Chalendar, E. Fricain, and D. Timotin, *Functional models and asymptotically orthonormal sequences*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 5, 1527–1549.

[6] Sun Yung A. Chang, *A characterization of Douglas subalgebras*, Acta Math. **137** (1976), no. 2, 82–89.

[7] Konstantin Dyakonov and Artur Nicolau, *Free interpolation by nonvanishing analytic functions*, Trans. Amer. Math. Soc. **359** (2007), no. 9, 4449–4465.

[8] J. P Earl, *On the interpolation of bounded sequences by bounded functions*, J. London Math. Soc. **2** (1970), 544–548.

[9] Pamela Gorkin and Raymond Mortini, *Asymptotic interpolating sequences in uniform algebras*, J. London Math. Soc. (2) **67** (2003), no. 2, 481–498.

[10] Pamela Gorkin, Sandra Pott, and Brett Wick, *Thin Sequences and Their Role in H^p Theory, Model Spaces, and Uniform Algebras*, Revista Matemática Iberoamericana (to appear).

[11] Håkan Hedenmalm, *Thin interpolating sequences and three algebras of bounded functions*, Proc. Amer. Math. Soc. **99** (1987), no. 3, 489–495, DOI 10.2307/2046351.

[12] Peter W. Jones, *Recent advances in the theory of Hardy spaces*, Proc. Amer. Math. Soc. (to appear.)

[13] P. and Rochberg Lopatto R., *Schatten-class truncated Toeplitz operators*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1984, pp. 829–838.

[14] Donald E. Marshall, *Subalgebras of L^∞ containing H^∞*, Acta Math. **137** (1976), no. 2, 91–98.

[15] Raymond Mortini, *Thin interpolating sequences in the disk*, Arch. Math. (Basel) **92** (2009), no. 5, 504–518, DOI 10.1007/s00013-009-3057-x.

[16] Donald Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286–299.

[17] H.S. Shapiro and A.L. Shields, *On some interpolation problems for analytic functions*, American J. Math. **83** (1961), 513–532.

[18] Carl and Wolff Sundberg Thomas H., *Interpolating sequences for QA_B*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 551–581, DOI 10.2307/1999068.

[19] A. L. Vol′berg, *Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason*, J. Operator Theory **7** (1982), no. 2, 209–218.

[20] T. Wolff, *Some theorems on vanishing mean oscillation*, Ph.D. Thesis, University of California at Berkeley (1979).

Commuting Dilations and Linear Positivstellensätze

Igor Klep, University of Auckland

Based on joint work with Bill Helton, Scott McCullough, and Markus Schweighofer

Summary. An operator C on a Hilbert space $\mathcal{H}$ *dilates* to an operator T on a Hilbert space $\mathcal{K}$ if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $C = V^*TV$. A main result presented in this talk is, for a positive integer d, the simultaneous dilation, up to a sharp factor $\vartheta(d)$, of all $d \times d$ symmetric matrices of operator norm at most

one to a collection of *commuting* self-adjoint contraction operators on a Hilbert space. Asymptotically $\vartheta(d)$ behaves as $\frac{\sqrt{d\pi}}{2}$, and for even $d \in \mathbb{N}$ we have

$$\vartheta(d) = \sqrt{\pi}\ \frac{\Gamma\left(\frac{d}{4}+1\right)}{\Gamma\left(\frac{d}{4}+\frac{1}{2}\right)}.$$

New probabilistic results for the binomial and beta distributions are needed to derive this analytic formula for $\vartheta(d)$.

Dilating to commuting operators has consequences for the theory of linear matrix inequalities (LMIs). Given a tuple $A = (A_1, \dots, A_g)$ of $\nu \times \nu$ symmetric matrices, the symmetric matrix polynomial

$$L(x) := I - \sum A_j x_j$$

is a *monic linear pencil* of size ν. The solution set $\mathbb{S}_L$ of the corresponding linear matrix inequality, consisting of those $x \in \mathbb{R}^g$ for which $L(x) \succeq 0$, is a *spectrahedron.* The set $\mathfrak{C}_L^{(g)}$ of tuples $X = (X_1, \dots, X_g)$ of symmetric matrices (of the same size) for which $L(X) := I - \sum A_j \otimes X_j$ is positive semidefinite, is a *free spectrahedron.* (Here $\otimes$ denotes the (Kronecker) tensor product of matrices.)

It is shown that any tuple X of $d \times d$ symmetric matrices in a bounded free spectrahedron $\mathcal{D}_L$ dilates, up to a scale factor depending only on d, to a tuple T of *commuting* self-adjoint operators with joint spectrum in the corresponding spectrahedron $\mathbb{S}_L$. The scale factor measures the extent that a positive map can fail to be completely positive.

Given another monic linear pencil $\tilde{L}$, the inclusion $\mathcal{D}_L \subset \mathcal{D}_{\tilde{L}}$ obviously implies the inclusion $\mathbb{S}_L \subset \mathbb{S}_{\tilde{L}}$ and thus can be thought of as its free relaxation. Determining if one free spectrahedron contains another can be done by solving an explicit LMI and is thus computationally tractable. The scale factor for commutative dilation of $\mathcal{D}_L$ gives a precise measure of the worst case error inherent in the free relaxation, over all monic linear pencils $\tilde{L}$ of size d.

The set $\mathfrak{C}^{(g)}$ of g-tuples of symmetric matrices of norm at most one is an example of a free spectrahedron known as the free cube and its associated spectrahedron is the cube $[-1,1]^g$. Interpreted in the general framework of this talk, Ben-Tal and Nemirovski treated the NP-hard cube inclusion problem $[-1,1]^g \subset \mathbb{S}_L$ by relaxing it to the free cube inclusion problem $\mathfrak{C}^{(g)} \subset \mathcal{D}_L$; they also bounded the error of this relaxation. The simultaneous dilation approach applied to the cube gives the sharp bound $\vartheta(d)$ and proves it equals theirs.

References

[B-TN02] A. Ben-Tal, A. Nemirovski: On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty, *SIAM J. Optim.* **12** (2002) 811–833.

[HKM12] J.W. Helton, I. Klep, S. McCullough: The convex Positivstellensatz in a free algebra, *Adv. Math.* **231** (2012) 516–534. (this article succeeds [HKM13] but appeared earlier)

[HKM13] J.W. Helton, I. Klep, S. McCullough: The matricial relaxation of a linear matrix inequality, *Math. Program.* **138** (2013) 401–445. (this article precedes [HKM12] but appeared later)

[HKMS] J.W. Helton, I. Klep, S. McCullough, M. Schweighofer: Dilations, Linear Matrix Inequalities, the Matrix Cube Problem and Beta Distributions, *preprint*, `http://arxiv.org/abs/1412.1481`

Integrability and regularity of rational functions

Greg Knese, Washington University

A basic question, which does not seem to have been answered systematically in the past, is: When is a rational function p/q in $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{T}^n)$? Here $\mathbb{T}$ is the unit circle in $\mathbb{C}$.

This is essentially about how the zeros of p and $q \in \mathbb{C}[z_1, \dots, z_n]$ compete in $\mathbb{R}^n$ or $\mathbb{T}^n$. Polynomials with such singularities appear in a variety of extremal problems in complex analysis and there is also some interest in the engineering literature. The paper [6] includes a detailed study of the integrability and regularity properties of $f = q/p$ on $\mathbb{T}^2$ where $q \in \mathbb{C}[z_1, z_2]$ and p is the specific polynomial $p(z_1, z_2) = 2 - z_1 - z_2$. The approach involves detailed power series computations and so our goal was to understand such examples more systematically.

Let $p \in \mathbb{C}[z_1, z_2]$ have bidegree (n, m) and define

$$\tilde{p}(z_1, z_2) = z_1^n z_2^m \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}.$$

We say p is *scattering stable* if $p(z_1, z_2) \neq 0$ for $|z_1|, |z_2| < 1$ and if p and $\tilde{p}$ have no common factors. The assumption of no zeros in $\mathbb{D}^2$ provides us with the analyticity we need to prove our results while the assumption of no common factors with $\tilde{p}$ is for convenience (to avoid easily resolved technicalities).

Our first main result is that for p scattering stable, we can give a concrete list of generators for the ideal

$$I_p = \{q \in \mathbb{C}[z_1, z_2] : q/p \in L^2(\mathbb{T}^2)\}.$$

The generators come from certain orthogonal complements of finite dimensional subspaces of $L^2(\frac{d\sigma}{|p|^2})$, where $d\sigma$ is Lebesgue measure on $\mathbb{T}^2$.

The second main result is a dimension count of truncations of I_p. Define

$$P_{j,k} = \{q \in I_p : \deg q \leq (j, k)\}.$$

Let $N_{\mathbb{T}^2}(p, \tilde{p})$ be the number of common zeros of p and $\tilde{p}$ on $\mathbb{T}^2$, where zeros must be counted with appropriate multiplicities as in Bezout's theorem. Then

$$\dim P_{j,k} = (j+1)(k+1) - \frac{1}{2} N_{\mathbb{T}^2}(p, \tilde{p}).$$

The proof involves studying the generalized eigenspace decomposition of a certain pair of commuting contractive operators on $P_{j,k}$ in $L^2(\frac{d\sigma}{|p|^2})$.

This work appears in [7], which is on the arXiv, but it draws from and is inspired by the other works in the references below.

It would be interesting of course to remove the assumption that p has no zeros in the bidisk and come up with an essentially algebraic approach to these problems.

References

[1] Joseph A. Ball, Cora Sadosky, and Victor Vinnikov, *Scattering systems with several evolutions and multidimensional input/state/output systems*, Integral Equations Operator Theory **52** (2005), no. 3, 323–393, DOI 10.1007/s00020-005-1351-y. MR2184571 (2006h:47013)

[2] Kelly Bickel, *Fundamental Agler decompositions*, Integral Equations Operator Theory **74** (2012), no. 2, 233–257, DOI 10.1007/s00020-012-1992-6. MR2983064

[3] Kelly Bickel and Greg Knese, *Inner functions on the bidisk and associated Hilbert spaces*, J. Funct. Anal. **265** (2013), no. 11, 2753–2790, DOI 10.1016/j.jfa.2013.08.002. MR3096989

[4] Brian J. Cole and John Wermer, *Ando's theorem and sums of squares*, Indiana Univ. Math. J. **48** (1999), no. 3, 767–791, DOI 10.1512/iumj.1999.48.1716. MR1736979 (2000m:47014)

[5] Jeffrey S. Geronimo and Hugo J. Woerdeman, *Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables*, Ann. of Math. (2) **160** (2004), no. 3, 839–906, DOI 10.4007/annals.2004.160.839. MR2144970 (2006b:42036)
[6] Dennis Goodman, *Some stability properties of two-dimensional linear shift-invariant digital filters*, IEEE Trans. Circuits and Systems **CAS-24** (1977), no. 4, 201–208. MR0444266 (56 #2624)
[7] Greg Knese, *Integrability and regularity of rational functions*, 2015, submitted.
[8] ———, *Polynomials with no zeros on the bidisk*, Anal. PDE **3** (2010), no. 2, 109–149, DOI 10.2140/apde.2010.3.109. MR2657451 (2011i:42051)
[9] Anton Kummert, *Synthesis of two-dimensional lossless m-ports with prescribed scattering matrix*, Circuits Systems Signal Process. **8** (1989), no. 1, 97–119, DOI 10.1007/BF01598747. MR998029 (90e:94048)

The Nullstellensatz and the Corona Theorem

Hyun-Kyoung Kwon, University of Alabama

Let $p_1, p_2, \cdots, p_m \in \mathbb{C}[z_1, z_2, \cdots, z_n]$ and denote by $V(p_1, p_2, \cdots, p_m)$ the set

$$V(p_1, p_2, \cdots, p_m) := \{z \in \mathbb{C}^n : p_i(z) = 0, \forall 1 \leq i \leq m\}.$$

We then have the following weak form of the Nullstellensatz [5]:

Theorem 1 (Hilbert's Weak Nullstellensatz)**.**

$$V(p_1, p_2, \cdots, p_m) = \emptyset$$

is equivalent to the existence of

$$q_1, q_2, \cdots, q_m \in \mathbb{C}[z]$$

with

$$p_1q_1 + p_2q_2 + \cdots p_mq_m = 1.$$

The question of interest now is to determine a bound for the degrees of the polynomials q_i whose existence are guaranteed by the Nullstellensatz. There are a number of results in this direction including the bounds obtained by W. D. Brownawell, G. Hermann, J. Kollar, and M. Sombra [2], [4], [7], [11]. The speaker, in joint work with A. Netaynun and T. T. Trent, uses some well-known methods for tackling the corona problem ([1], [6], [10], [12], [13]) to recover in a simple way the bound given by W. D. Brownawell in the special case when the p_i are all univariate polynomials (n=1). More precisely, we have the following result [8]:

Theorem 2. *Let $p_j \in \mathbb{C}[z]$ be m polynomials with no common zero in $\mathbb{C}$ and let $\deg p_j = d_j$, where $d_1 \geq d_2 \geq \cdots \geq d_m$. Then there exist m polynomials $q_j \in \mathbb{C}[z]$ such that*

$$\sum_j p_jq_j = 1$$

with the bound

$$\deg q_j \leq d_1 - 1.$$

One can go even further and obtain a bound for the highest degree of the polynomial entries of an inverse matrix to a matrix of polynomials satisfying the corona condition [9]:

Theorem 3. *Let $F(z) = [p_{ij}(z)]$ be an $m \times n$ matrix of polynomials $p_{ij}(z) \in \mathbb{C}[z]$ satisfying the corona condition*

$$F(z)F(z)^* \geq \epsilon > 0,$$

for all $z \in \mathbb{C}$. We can assume without loss of generality that the k_i, the maximum degree of the entries in the i-th row of $F(z)$, are in increasing order. Then there exists $G(z) = [q_{ij}(z)], q_{ij}(z) \in \mathbb{C}[z]$, an $n \times m$ matrix of polynomials such that $F(z)G(z) = I_m$. Moreover, if $1 \leq k_1$, then the degree bound for the $q_{ij}(z)$ is given by

$$(2m-1)k_m + 2k_{m-1} + 4k_{m-2} + \cdots + 2(m-1)k_1 - m.$$

Otherwise, if l is the smallest positive integer such that $1 \leq k_{l+1}$, then the degree bound for the $q_{ij}(z)$ is given by

$$2k_m + 4k_{m-1} + \cdots + 2(m-l)k_{l+1} - (m-l).$$

Whether one can use the methods employed to obtain the two previous results in the multivariate polynomial situation remains open.

References

[1] M. Andersson, *Topics in Complex Analysis,* Springer-Verlag, New York, 1996.

[2] W. D. Brownawell, *Bounds for the degrees in the Nullstellensatz,* Ann. of Math. (2) **126** (1987), 577-592.

[3] L. Carleson, *Interpolation by bounded analytic functions and the corona problem,* Ann. of Math. (2) **76** (1962), 547-559.

[4] G. Hermann, *Die Frage der endlich vielen Schritte in der Theorie der Polynomideale,* Math. Ann. **95** (1926), 736-788.

[5] D. Hilbert, *Über die vollen Invariantesysteme,* Math. Ann. **42** (1893), 313-373.

[6] L. Hörmander, *An Introduction to Complex Analysis in Several Variables,* van Nostrand, Princeton, 1966.

[7] J. Kollár, *Sharp effective Nullstellensatz,* J. Amer. Math. Soc. (4) **1** (1988), 963-975.

[8] H. Kwon, A. Netyanun, and T. T. Trent, *An analytic approach to the degree bound in the Nullstellensatz*, accepted to Proc. Amer. Math. Soc.

[9] H. Kwon, A. Netyanun, and T. T. Trent, *An Estimate for the Degree Bound of a Matrix of Polynomials*, submitted.

[10] J. Ryle and T. T. Trent, *A corona theorem for certain subalgebras of $H^\infty(\mathbb{D})$ II*, Houston J. Math. (4) **38** (2012), 1277-1295.

[11] M. Sombra, *A sparse effective Nullstellensatz,* Adv. Appl. Math. (2) **22** (1999), 271-295.

[12] T. T. Trent, *A new estimate for the vector-valued corona problem,* J. Funct. Anal. **189** (2002), 267-282.

[13] T. T. Trent and X. Zhang. *A matricial corona theorem,* Proc. Amer. Math. Soc. (9) **134** (2006), 2549-2558.

Non-commutative Functional Calculus and Spectral Theory

John E. McCarthy[3], Washington University, St. Louis

joint with Jim Agler[4], U.C. San Diego

[3]Partially supported by National Science Foundation Grant DMS 1300280

[4]Partially supported by National Science Foundation Grant DMS 1361720

In [3], J.L. Taylor considered the problem of developing a non-commutative functional calculus for d-tuples in $\mathrm{Ł}(X)$, the bounded linear operators on a Banach space X. His idea was to start with the algebra $\mathbb{P}^d$, the algebra of free polynomials in d variables over the complex numbers, and consider what he called "satellite algebras", that is algebras $\mathcal{A}$ that contained $\mathbb{P}^d$, and with the property that every representation from $\mathbb{P}^d$ to $\mathrm{Ł}(X)$ that extends to a representation of $\mathcal{A}$ has a unique extension. As a representation of $\mathbb{P}^d$ is determined by choosing the images of the generators, *i.e.* choosing $T = (T^1, \ldots, T^d) \in \mathrm{Ł}(X)^d$, the extension of the representation to $\mathcal{A}$, when it exists, would constitute an $\mathcal{A}$-functional calculus for T. The class of satellite algebras that Taylor considered, which he called free analytic algebras, were intended to be non-commutative generalizations of the algebras $O(U)$, the algebra of holomorphic functions on a domain U in $\mathbb{C}^d$ (and indeed he proved in [3, Prop 3.3] that when $d = 1$, these constitute all the free analytic algebras).

We develop a non-commutative functional calculus using the algebras $H^\infty(G_\delta)$, the bounded nc functions (see [2] for a treatment of nc-functions) on some polynomial polyhedron G_δ, the set of all d-tuples of matrices x satisfying $\|\delta(x)\| < 1$ for some matrix δ with entries in $\mathbb{P}^d$.

Our treatment relies on a model theory for $H^\infty(G_\delta)$ developed in [1].

References

[1] J. Agler and J.E. McCarthy. Global holomorphic functions in several non-commuting variables. *Canad. J. Math.*, 67(2):241–285, 2015.

[2] Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov. *Foundations of free non-commutative function theory.* AMS, Providence, 2014.

[3] J.L. Taylor. Functions of several non-commuting variables. *Bull. Amer. Math. Soc.*, 79:1–34, 1973.

Matrix Concomitants, Schemes, and Azumaya Algebras

Paul S. Muhly, University of Iowa
joint with Erin Griesenauer and Baruch Solel, Technion

Throughout, G will denote the projective linear group, $PGL(n, \mathbb{C})$, viewed as the automorphism group of the $n \times n$ matrices, $M_n(\mathbb{C})$, and K will denote $PU(n, \mathbb{C})$, viewed as the group of $*$-preserving automorphisms of $M_n(\mathbb{C})$. If H is an algebraic group acting algebraically on an algebraic variety X and if $\rho : H \to G$ is an algebraic representation H in G, then a *matrix concomitant* on X is simply a rational map $F : X \to M_n(\mathbb{C})$ such that $F(xh) = \rho(h) \cdot F(x)$, $h \in H$, $x \in X$. Matrix concomitants have had a long history in algebra, particularly in invariant theory and they have become a centerpiece in the nascent field of noncommutative algebraic geometry. For a comprehensive introduction to this theory that is also useful for our purposes see [6]. Matrix concomitants arise naturally also in the new field of noncommutative function theory. Indeed, one of the defining properties of a noncommutative function as presented in the important recent monograph [4] is that it be a (sequence of) matrix concomitant(s). We are interested specifically in the holomorphic matrix concomitants defined on all d-tuples of $n \times n$ matrices $M_n(\mathbb{C})^d$. That is, we want to analyze the holomorphic functions $F : M_n(\mathbb{C})^d \to M_n(\mathbb{C})$ such that $F(s^{-1}\mathfrak{z}s) = s^{-1}F(\mathfrak{z})s$, for $s \in G$ and $\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d) \in M_n(\mathbb{C})^d$. We denote this space by $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$.

We appeal to important work of Procesi [8,9]. Let $\mathbb{I}_0(d,n)$ denote all the polynomial functions $p : M_n(\mathbb{C}) \to \mathbb{C}$ such that $p(s^{-1}\mathfrak{z}s) = p(\mathfrak{z})$, i.e., all the polynomial invariants. Then $\mathbb{I}_0(d,n)$ is a finitely generated subalgebra of all polynomial functions on $M_n(\mathbb{C})^d$, $\mathbb{C}[M_n(\mathbb{C})^d]$, by Hilbert's famous theorem [3]. If $Q(d,n)$ denotes the spectrum of $\mathbb{I}_0(d,n)$, then $Q(d,n)$ is an abstract algebraic variety and the inclusion $\mathbb{I}_0(d,n)$ in $\mathbb{C}[M_n(\mathbb{C})^d]$ induces an algebraic projection π of $M_n(\mathbb{C})^d$ onto $Q(d,n)$. Thanks to work of Artin and Procesi, [1, Theorem 12.6] and [8, Theorem 4.1], $Q(d,n)$ is the categorical quotient of G acting on $M_n(\mathbb{C})^d$. Further, if $\mathcal{V}(d,n)$ denotes $\{\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d) \mid \text{the } Z_i \text{ generate } M_n(\mathbb{C})\}$, then $\mathcal{V}(d,n)$ is Zariski dense in $M_n(\mathbb{C})^d$, $\pi(\mathcal{V}(d,n)) := Q_0(d,n)$ is contained in the smooth points of $Q(d,n)$, and $(\mathcal{V}(d,n), \pi, Q_0(d,n))$ has the structure of holomorphic principal G-bundle over $Q_0(d,n)$ [8, Theorem 5.10]. We write $\mathfrak{M}_n$ for the associated $M_n(\mathbb{C})$-fibre bundle.

Theorem 1. *$Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ is naturally isomorphic to the holomorphic cross sections of $\mathfrak{M}_n$, $\Gamma_h(Q_0(d,n), \mathfrak{M}_n)$.*

Theorem 1 allows us to view $Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^G$ as a space of "nice functions" on its space of irreducible representations. It also suggests an obvious sheaf with which to study local properties of holomorphic matrix concomitants. It is somewhat different from the one proposed by Luminet in [7]. The differences need further study.

We are interested in the *continuous cross sections* of $\mathfrak{M}_n$ as well as the holomorphic ones. One problem we face is that $\mathfrak{M}_n$ does not have an evident natural $*$-structure. However we may impose one by choosing a (topological) reduction of $(\mathcal{V}(d,n), \pi, Q_0(d,n))$ to a principal K-bundle. This may be done in many ways and they all yield $*$-isomorphic topological bundles, but the holomorphic features become "scrambled" in a sense which we leave imprecise here. One promising reduction is based on the following notion.

Definition 1. A point $\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d)$ in $M_n(\mathbb{C})^d$ is called *hypernormal* in case $\sum_{i=1}^d [Z_i, Z_i^*] = 0$, where the brackets denote commutators. We write $\mathcal{HN}(d,n)$ (resp. $\mathcal{HN}_0(d,n)$) for all the hypernormal points (resp. the irreducible hypernormal points).

Hypernormal points come about naturally by applying symplectic reduction, also known as Kempf-Ness theory [5], to the G-action on $\mathcal{V}(d,n)$.

Theorem 2. [6, Theorem 2.11] *$\mathcal{HN}(d,n)$ is K-invariant and $\mathcal{HN}(d,n)/K$ is naturally identified with $Q(d,n)$. Further, $\mathcal{HN}_0(d,n)$ is a reduction of $\mathcal{V}(d,n)$ to a principal K-bundle over $Q_0(d,n)$.*

Let $\mathfrak{M}_n^*$ be the bundle over $Q_0(d,n)$ whose bundle space is $\mathcal{HN}_0(d,n) \times_K M_n(\mathbb{C})$. Then $\mathfrak{M}_n$ and $\mathfrak{M}_n^*$ are topologically isomorphic. Let $\mathcal{D}$ be a domain in $Q_0(d,n)$ whose closure $\overline{\mathcal{D}}$ is a Stein compact subset of $Q_0(d,n)$. Let $\partial\mathcal{D}$ be the Shilov boundary of $\mathcal{D}$ and let $A(\partial\mathcal{D}, \mathfrak{M}_n^*)$ be the closure of $\Gamma_h(\overline{\mathcal{D}}, \mathfrak{M}_n)$ in the continuous sections of $\mathfrak{M}_n^*$, $\Gamma_c(\partial D, \mathfrak{M}_n^*)$. The following theorem is the main result of our talk.

Theorem 3. *Each point of the Choquet boundary of $\mathcal{D}$ gives (through evaluation) a boundary representation of $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}_n^*)$ for $A(\partial\mathcal{D}, \mathfrak{M}_n^*)$ in the sense of Arveson* [2]*, and so $\Gamma_c(\partial\mathcal{D}, \mathfrak{M}_n^*)$ is the C^*-envelope of $A(\partial\mathcal{D}, \mathfrak{M}_n^*)$. Further, $A(\partial\mathcal{D}, \mathfrak{M}_n^*)$ is an Azumaya algebra over its center, which is the closure in $C(\partial\mathcal{D})$ of the functions that are holomorphic in a neighborhood of $\overline{\mathcal{D}}$.*

References

[1] M. Artin. On Azumaya algebras and finite dimensional representations of rings. *J. Algebra*, 11:532–563, 1969.
[2] William B. Arveson. Subalgebras of C^*-algebras. *Acta Math.*, 123:141–224, 1969.
[3] David Hilbert. Ueber die Theorie der algebraischen Formen. *Math. Ann.*, 36(4):473–534, 1890.
[4] Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov. *Foundations of free noncommutative function theory*, volume 199 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.
[5] George Kempf and Linda Ness. The length of vectors in representation spaces. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 233–243. Springer, Berlin, 1979.
[6] Lieven Le Bruyn. *Noncommutative geometry and Cayley-smooth orders*, volume 290 of *Pure and Applied Mathematics (Boca Raton)*. Chapman & Hall/CRC, Boca Raton, FL, 2008.
[7] Denis Luminet. Functions of several matrices. *Boll. Un. Mat. Ital. B (7)*, 11(3):563–586, 1997.
[8] Claudio Procesi. Finite dimensional representations of algebras. *Israel J. Math.*, 19:169–182, 1974.
[9] Claudio Procesi. The invariant theory of $n \times n$ matrices. *Advances in Math.*, 19(3):306–381, 1976.

On Operator Algebras associated with Monomial Ideals in Noncommuting Variables

Orr Moshe Shalit, Technion
joint with Evgenios Kakariadis, U. Newcastle

This report is a brief report on the very long paper [2] (joint with Kakariadis). The work combines various aspects of multivariable operator theory (noncommutative varieties in the sense of Popescu [5], C*-algebras and non-selfadjoint algebras associated to C*-correspondences in sense of Pimsner [4] and Muhly-Solel [3], operator algebras arising from subproduct systems in the sense of Shalit-Solel [6] — some of which turn out to be precisely the bounded free NC functions, in the sense of Ball, M^cCarthy, Vinnikov and others, on noncommutative varieities), and continues the classification of universal algebras studied in [1] by Davidson-Ramsey-Shalit. We obtain complete and detailed results regarding the structure, properties, interelations and classification of the different operator algebras which one may associate with monomial ideals. One key novelty of our work is that we use C*-correspondence techniques to study algebras which are not given as operator algebras arising form C*-correspondences.

Fix an orthonormal basis $\{e_1, \dots, e_d\}$ for $\mathbb{C}^d$ and write $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_n}$ for every word $\mu = \mu_1 \dots \mu_n \in \mathbb{F}_+^d$. Given a monomial ideal $\mathcal{I}$ in $\mathbb{C}\langle x_1, \dots, x_d\rangle$ let $X = (X(n))$ be the associated subproduct system [6]. Let us write $\mathcal{F}_X = \oplus_{n \geq 0} X(n)$ and let the shift operators $\{T_i\}_{i=1}^d$ defined by

$$T_i e_\nu = \begin{cases} e_{i\nu} & \text{if } i\nu \in \Lambda^*, \\ 0 & \text{otherwise.} \end{cases}$$

The C*-algebras

$$\mathrm{C}^*(T) := \mathrm{C}^*(I, T_i | i = 1, \dots, d) \quad \textit{and} \quad \mathrm{C}^*(T)/\mathcal{K}(\mathcal{F}_X)$$

are the *Toeplitz* and the *Cuntz* algebra of X. (Our Cuntz algebras contain and generalize well known classes such as Cuntz-Krieger C*-algebras and Matsumoto's subshift C*-algebras). The nonselfadjoint subalgebra

$$\mathcal{A}_X := \overline{\mathrm{alg}}\{I, T_i | i = 1, \dots, d\}$$

of $C^*(T)$ is the *tensor algebra* of X. (The tensor algebras of subproduct systems are a rich class of algebras containing the disc algebra, Popescu's noncommutative disc algebras, as well as the "continuous" multipliers on Drury-Arveson space).

The family $\{T_i\}_{i=1}^d$ satisfies a number of properties; for example it is orthogonal and $T_\mu^* T_\mu T_i = T_i T_{\mu i}^* T_{\mu i}$. Hence the linear space

$$E = \overline{\mathrm{span}}\{T_i a | a \in A, i = 1, \dots, d\},$$

becomes a C*-correspondence over the commutative unital C*-algebra A generated by $T_\mu^* T_\mu$, where μ runs over all the set of all *allowed words* Λ^* (i.e., powers μ such that $x^\mu \notin \mathcal{I}$). Consequently we obtain the *Toeplitz-Pimsner algebra* $\mathcal{T}_E$, the *Cuntz algebra* $\mathcal{O}_E$, and the *tensor algebra* $\mathcal{T}_E^+$ in the sense of Muhly and Solel [3]. We denote the left action by ϕ_E, and refer to ${}_A E_A$ as *the C*-correspondence associated with the monomial ideal* $\mathcal{I}$.

We use techniques from C*-correspondences to study the operator algebras arising from subproduct systmems. We obtain a complete map of connections between the various C*-algebras arising. For example:

Theorem A. *Let ${}_A E_A$ be the C*-correspondence of a monomial ideal $\mathcal{I} \triangleleft \mathbb{C}\langle x_1, \dots, x_d\rangle$. Then the following diagrams hold:*

$$\begin{array}{ccc} & C^*(T) \not\simeq \mathcal{T}_E \Leftrightarrow \mathcal{I} \neq (0) \Leftrightarrow E \not\simeq \mathbb{C}^d & \\ \swarrow & & \searrow \\ \ker \phi_E \neq (0) & & \ker \phi_E = (0) \\ \Updownarrow & & \Updownarrow \\ \mathcal{O}_E \simeq C^*(T) & & \mathcal{O}_E \simeq C^*(T)/\mathcal{K}(\mathcal{F}_X) \end{array}$$

$$\begin{array}{c} C^*(T) \simeq \mathcal{T}_E \Leftrightarrow \mathcal{I} = (0) \Leftrightarrow E \simeq \mathbb{C}^d \\ \Downarrow \\ \mathcal{O}_E \simeq C^*(T)/\mathcal{K}(\mathcal{F}_X) \simeq \mathcal{O}_d\,, \ker \phi_E = (0) \end{array}$$

As a corollary, one immediately obtains that the Toeplitz and Cuntz algebras of X are nuclear, because this is known for $\mathcal{O}_E$ and and $\mathcal{T}_E$. We also compute the C*-envelopes of the tensor algebras. There is a dichotomy, depending on the structure of the ideal $\mathcal{I}$. For example:

Theorem B. *Let X be the subproduct system of a monomial ideal $\mathcal{I} \triangleleft \mathbb{C}\langle x_1, \dots, x_d\rangle$ of finite type and let $q \colon C^*(T) \to C^*(T)/\mathcal{K}(\mathcal{F}_X)$. Then the implications* (1), (2), *and* (3) *of the following diagrams hold:*

$$\begin{array}{c} q|_{\mathcal{A}_X} \text{ is not completely isometric} \\ \Updownarrow (1) \\ C^*_{env}(\mathcal{A}_X) \simeq C^*(T) \\ \Downarrow (3) \quad \Uparrow (4) \\ \forall i = 1, \dots, d, \exists \mu_i \in \Lambda^*. \mu_i i \notin \Lambda^* \end{array} \qquad \begin{array}{c} q|_{\mathcal{A}_X} \text{ is completely isometric} \\ \Updownarrow (2) \\ C^*_{env}(\mathcal{A}_X) \simeq C^*(T)/\mathcal{K}(\mathcal{F}_X) \\ \Downarrow (4) \quad \Uparrow (3) \\ \exists i \in \{1, \dots, d\}, \forall \mu \in \Lambda^*. \mu i \in \Lambda^* \end{array}$$

The implication (4) holds in many cases too.

The paper includes much more, including hyperrigidity, classification of the nonselfadjoint algebras, and the analysis of a special kind of dynamical system.

References

[1] K.R. Davidson, C. Ramsey and O.M. Shalit, *The isomorphism problem for some universal operator algebras*, Adv. Math. **228:1** (2011), 167–218.
[2] E.T.A. Kakariadis and O.M. Shalit, *On Operator Algebras associated with Monomial Ideals in Noncommuting Variables*, preprint arXiv:1501.06495 [math.OA].
[3] P.S. Muhly and B. Solel, *Tensor algebras over C*-correspondences: representations, dilations and C*-envelopes*, J. Funct. Anal. **158:2** (1998), 389–457.
[4] M.V. Pimsner, *A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* $\mathbb{Z}$, Free probability theory (Waterloo, ON, 1995), 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.
[5] G. Popescu, *Operator theory on noncommutative varieties*, Indiana Univ. Math. J. **5:2** (2006), 389–442.
[6] O.M. Shalit and B. Solel, *Subproduct Systems*, Doc. Math. **14** (2009), 801–868.

Holomorphic automorphisms of noncommutative polyballs

Gelu Popescu, University of Texas at San Antonio

Outline of the talk:

1. *Noncommutative polyballs and free holomorphic functions*
2. *Maximum principle and Schwarz type results*
3. *Holomorphic automorphisms of noncommutative polyballs*
4. *Automorphisms of Cuntz-Toeplitz algebras*
5. *Automorphisms of the polyball algebra* $\mathcal{A}(\mathbf{B_n})$ *and Hardy algebra* $H^\infty(\mathbf{B_n})$
6. *The automorphism group* $Aut(\mathbf{B_n})$ *and unitary projective representations*

Recently (see [8], [9]), we have tried to unify the multivariable operator model theory for ball-like domains and commutative polydiscs, and extend it to a more general class of noncommutative polydomains (which includes the regular polyballs) and use it to develop a theory of free holomorphic functions. What is remarkable for these polydomains is that they have universal operator models, in a certain sense, which are (weighted) creation operators acting on tensor products of full Fock spaces. The model theory and the free holomorphic function theory on these polydomains are related, via noncommutative Berezin transforms, to the study of the operator algebras generated by the universal models, as well as to the theory of functions in several complex variable ([4], [13]). It is the interplay between these three fields that lead to a rich analytic function theory on these noncommutative polydomains. Our work on curvature invariant [10] and Euler characteristic [11] on noncommutative regular polyballs has led us to study the free holomorphic automorphisms of these polyballs, which is the goal of the recent paper [12] and continues work of Voiculescu [15], of Davidson and Pitts [2], of Helton, Klep, McCullough and Singled [3], of Benhida and Timotin [1], and of the author in [6], [7].

The abstract regular polyball $\mathbf{B_n}$, $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, is a noncommutative analogue of the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_1})_1$, which has been recently studied in connection with operator model theory, curvature invariant, and Euler characteristic. The regular polyball $\mathbf{B_n}$ is a logarithmically convex complete Reinhardt noncommutative domain, in an appropriate sense, which is, arguably, the right noncommutative analogue of the commutative polydisc which has a natural operator model theory. In [12], we study free holomorphic functions on $\mathbf{B_n}$ and provide analogues of several classical results from complex analysis such as:

Abel theorem, Hadamard formula, Cauchy inequality, Schwarz lemma, and maximum principle. These results are used together with a class of noncommutative Berezin transforms to obtain a complete description of the group $\mathrm{Aut}(\mathbf{B_n})$ of all free holomorphic automorphisms of the polyball $\mathbf{B_n}$, which is an analogue of Rudin's characterization [13] of the holomorphic automorphisms of the polydisc, and show that

$$\mathrm{Aut}(\mathbf{B_n}) \simeq \mathrm{Aut}((\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_1})_1).$$

This resembles the classical result of Ligocka [5] and Tsyganov [14] concerning biholomorphic automorphisms of product spaces with nice boundaries. If $\mathbf{m} = (m_1, \ldots, m_q) \in \mathbb{N}^q$, we show that the polyballs $\mathbf{B_n}$ and $\mathbf{B_m}$ are free biholomorphic equivalent if and only if $k = q$ and there is a permutation σ such that $m_{\sigma(i)} = n_i$ for any $i \in \{1, \ldots, k\}$. This extends Poincaré's result [4] that the open unit ball of $\mathbb{C}^n$ is not biholomorphic equivalent to the polydisk $\mathbb{D}^n$, to our noncommutative setting.

The abstract polyball $\mathbf{B_n}$ has a universal model $\mathbf{S} := \{\mathbf{S}_{i,j}\}$ consisting of left creation operators acting on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ of full Fock spaces. We proved in [9] (in a more general setting) that $\mathbf{X}$ is a *pure* element in the regular polyball $\mathbf{B_n}(\mathcal{H})^-$ if and only if there is a Hilbert space $\mathcal{K}$ and a subspace $\mathcal{M} \subset F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{K}$ invariant under each operator $\mathbf{S}_{i,j} \otimes I$ such that $X_{i,j}^* = (\mathbf{S}_{i,j}^* \otimes I)|_{\mathcal{M}^\perp}$ under an appropriate identification of $\mathcal{H}$ with $\mathcal{M}^\perp$. The existence of the universal model plays an important role, since it will make the connection between noncommutative function theory, operator algebras, and complex function theory in several variables. The latter is due to the fact that the joint eigenvectors for the universal model are parameterized by the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$ via the Berezin transforms (see [8]). The noncommutative Hardy algebra $\mathbf{F}_\mathbf{n}^\infty$ (resp. the polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$) is the weakly closed (resp. norm closed) non-selfadjoint algebra generated by $\{\mathbf{S}_{i,j}\}$ and the identity. We prove that

$$\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_\mathbf{n}}(C^*(\mathbf{S})) \simeq \mathrm{Aut}_u(\boldsymbol{\mathcal{A}}_\mathbf{n}) \simeq \mathrm{Aut}_u(\mathbf{F}_\mathbf{n}^\infty) \simeq \mathrm{Aut}(\mathbf{B_n}),$$

where $\mathrm{Aut}_{\boldsymbol{\mathcal{A}}_\mathbf{n}}(C^*(\mathbf{S}))$ is the group of automorphisms of the Cuntz-Toeplitz C^*-algebra $C^*(\mathbf{S})$ which leaves invariant the noncommutative polyball algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$, and $\mathrm{Aut}_u(\boldsymbol{\mathcal{A}}_\mathbf{n})$ (resp. $\mathrm{Aut}_u(\mathbf{F}_\mathbf{n}^\infty)$) is the group of unitarily implemented automorphisms of the algebra $\boldsymbol{\mathcal{A}}_\mathbf{n}$ (resp. $\mathbf{F}_\mathbf{n}^\infty$)). Moreover, we obtain formulas for the elements of these automorphism groups in terms of noncommutative Berezin transforms. As a consequence, we obtain a concrete description for the group of automorphisms of the tensor product $\mathcal{T}_{n_1} \otimes \cdots \otimes \mathcal{T}_{n_k}$ of Cuntz-Toeplitz algebras which leave invariant the tensor product $\mathcal{A}_{n_1} \otimes_{min} \cdots \otimes_{min} \mathcal{A}_{n_k}$ of noncommutative disc algebras, which extends Voiculescu's result [15] when $k = 1$. In particular, each holomorphic automorphism of the regular polyball $\mathbf{B_n}$ induces an automorphism of the tensor product of Cuntz algebras $\mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_k}$ which leaves invariant the non-self-adjoint subalgebra $\mathcal{A}_{n_1} \otimes_{min} \cdots \otimes_{min} \mathcal{A}_{n_k}$. We also prove that the free holomorphic automorphism group $\mathrm{Aut}(\mathbf{B_n})$ is a σ-compact, locally compact topological group with respect to the topology induced by the metric

$$d_{\mathbf{B_n}}(\phi, \psi) := \|\phi - \psi\|_\infty + \|\phi^{-1}(0) - \psi^{-1}(0)\|, \qquad \phi, \psi \in \mathrm{Aut}(\mathbf{B_n}).$$

Using the structure of the free holomorphic automorphisms of the regular polyball $\mathbf{B_n}$, one can provide a concrete unitary projective representation of the topological group $\mathrm{Aut}(\mathbf{B_n})$, with respect to the metric $d_{\mathbf{B_n}}$, in terms of noncommutative Berezin kernels associated with regular polyballs.

We mention that the results presented in this talk (see also [12]) will be used in a future paper to study the structure of the automorphism groups associated with certain classes of noncommutative varieties in polyballs, including the case of commutative operatorial polyballs. We also expect some of our results to extend to more general noncommutative polydomains ([8], [9]).

References

[1] C. Benhida and D. Timotin, Automorphism invariance properties for certain families of multioperators, *Operator theory live*, **5-15**, Theta Ser. Adv. Math., **12**, Theta, Bucharest, 2010.

[2] K. R. Davidson and D. Pitts, The algebraic structure of non-commutative analytic Toeplitz algebras, *Math. Ann.* **311** (1998), 275–303.

[3] J.W. Helton, I. Klep, S. McCullough, and N. Slingled, Noncommutative ball maps, *J. Funct. Anal.* **257** (2009), 47–87.

[4] S.G. Krantz, *Function theory of several complex variables.* Reprint of the 1992 edition. AMS Chelsea Publishing, Providence, RI, 2001. xvi+564 pp.

[5] E. Ligocka, On proper holomorphic and biholomorphic mappings between product domains, *Bull.Acad. Polon.Sci., Ser. Sci. Math.* **28** (1980), 319–323.

[6] G. Popescu, Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$, *J. Reine Angew. Math.* **638** (2010), 119–168.

[7] G. Popescu, Free biholomorphic classification of noncommutative domains *Int. Math. Res. Not.* IMRN **2011**, no. 4, 784–850.

[8] G. Popescu, Berezin transforms on noncommutative varieties in polydomains, *J. Funct. Anal.* **265** (2013), no. 10, 2500–2552.

[9] G. Popescu, Berezin transforms on noncommutative polydomains, *Trans. Amer. Math. Soc.*, in press.

[10] G. Popescu, Curvature invariant on noncommutative polyballs, *Adv. Math.*, in press.

[11] G. Popescu, Euler characteristic on noncommutative polyballs, *J. Reine Angew. Math.*, in press.

[12] G. Popescu, Holomorphic automorphisms of noncommutative polyballs, submitted.

[13] W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam 1969 vii+188 pp.

[14] Sh. Tsyganov, Biholomorphic maps of the direct product of domains, *Mat. Zametki.* **41** (1987), no.6, 824–828. (Russian and English translation)

[15] D. Voiculescu, Symmetries of some reduced free product C^*-algebras, *Lecture Notes in Math.* **1132**, 556–588, Springer Verlag, New York, 1985.

Dixmier trace of quotient module on bounded symmetric domain

Kai Wang, Tübingen University
joint with Harald Upmeier

The Dixmier trace of Hilbert space operators is of fundamental importance in geometric analysis and quantization theory, for example for pseudodifferential operators on compact manifolds, or for Hankel and Toeplitz operators on strongly pseudoconvex domains [1, 3, 3]. More recently, substantial results concerning the Dixmier trace have been obtained for suitable quotient modules of Hardy space [5, 7], revealing a close connection to the Hilbert module program in operator theory.

In what follows we are concerned with 'restricted' Toeplitz type operators on hermitian bounded symmetric domains $D = G/K$ of arbitrary rank r. In order to describe the results explicitly, let S be the Shilov boundary of D. Since K acts transitively on S, there exists a unique $K-$invariant probability measure ds on S. Denote $L^2(S)$ be the space of L^2-integrable functions, and define the *Hardy space*

$$H^2(S) = \{\phi \in L^2(S) : \phi \text{ holomorphic on } D\}.$$

In previous work [6] it was shown that Toeplitz operators T_f with smooth symbol function $f \in C^\infty(S)$ is not essentially commutative (if $r > 1$). In order to obtain a suitable Hilbert quotient module we consider the 'determinantal' variety defined by

$$Z^1 := \{z \in Z : rank(z) \leq 1\}.$$

The unit ball $D \cap Z^1$ is a strongly-pseudoconvex domain (singular at the origin), with a K-homogeneous smooth boundary

$$S_1 = \partial(D \cap Z^1).$$

Denote $\mathcal{I}(Z^1) = \{\phi \in H^2(S) : \phi(z) = 0\ \forall z \in Z^1 \cap D\}$, $H_1^2(S) = H^2(S) \ominus \mathcal{I}(Z^1)$ be the submodule and quotient module, respectively. For smooth symbols $f \in C^\infty(S)$, define

$$S_f := P_1\, T_f\, P_1,$$

where P_1 is the orthogonal projection onto $H_1^2(S)$.

Let $n = \dim Z^1$. Our first main result shows that n-fold products of commutators of such operators belong to the Macaev class $\mathcal{L}^{1,\infty}$.

Theorem 1. *For real polynomials $f_1, g_1, \cdots, f_n, g_n$, we have that*

$$[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] \in \mathcal{L}^{1,\infty}.$$

For $T \in \mathcal{L}^{1,\infty}$ the *Dixmier trace*, denoted by $tr_\omega(T)$, depends a priori on a choice of positive functional ω on $l^\infty(\mathbb{N})$ vanishing on $c_0(\mathbb{N})$. For the so-called *measurable* operators T the value $tr_\omega(T)$ is independent of ω. More precisely, for a positive operator T,

$$tr_\omega(T) = \lim_{N\to\infty} \frac{\sum_{i=1}^N \mu_i(T)}{\log N},$$

whenever the limit exists; here $\{\mu_i(T)\}_i$ is the singular value sequence of T. We refer the reader to [2] for more details.

Theorem 2. *In the setting of the above Theorem, we have that*

$$tr_\omega[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] = \frac{1}{n!(2\pi)^n} \int_{S_1} \prod_j \mathcal{L}(\overline{\partial_b}\, \widehat{f}_j, \partial_b \widehat{g}_j)\eta \wedge (d\eta)^{n-1},$$

where $\widehat{f}$ is the harmonic extension of f restricted to S_1, ∂_b the boundary ∂-operator, $\mathcal{L}$ the dual of the Levi form on the anti-holomorphic tangent bundle and η the definition function for S_1.

With little effort, one sees that the integral in the right side makes sense even for smooth symbol. It raises the following problem:

Problem 1. *prove the above theorems for the smooth symbols.*

This is already challenging for the basic case of rank 2-domains (pseudo-differential operators on spheres).

References

[1] A. Connes, *The action functional in noncommutative geometry*, Comm. Math. Phys., 117 (1988), 673-683.
[2] A. Connes, *Non-commutative Geometry*, Academic Press, San Diego, 1994.
[3] M. Englis, K. Guo and G. Zhang, *Toeplitz and Hankel operators and Dixmier traces on the unit ball of* $\mathbb{C}^n$, Proc. Amer. Math. Soc. 137 (2009), 3669-3678.
[4] M. Englis and G. Zhang, *Hankel operators and the Dixmier trace on strictly pseudoconvex domains*, Documenta Math., 15 (2010), 601-622.
[5] K. Guo, K. Wang and G. Zhang, *Trace formulas and p-essentially normal properties of quotient modules on the bidisk*, 67 (2012), 511-535.
[6] H. Upmeier, *Toeplitz C^*-algebras on bounded symmetric domains*, Ann. Math., 119(1984), 549-576.
[7] H. Upmeier and K. Wang, *Dixmier Trace for Toeplitz Operators on Symmetric Domains*, preprint.

Commutators in the Two-Weight Setting

Brett D. Wick, Washington University
joint with Irina Holmes and Michael T. Lacey, Georgia Tech.

1. Statement of Main Results

Let μ be a weight on $\mathbb{R}$, i.e. function that is positive almost everywhere and is locally integrable. Then define $L^2(\mathbb{R};\mu) \equiv L^2(\mu)$ to be the space of functions which are square integrable with respect to the measure $\mu(x)dx$, namely

$$\|f\|^2_{L^2(\mu)} \equiv \int_{\mathbb{R}} |f(x)|^2 \mu(x)dx < \infty.$$

For an interval I, let $\langle \mu \rangle_I \equiv \frac{1}{|I|}\int_I \mu(x)dx$. And, similarly, set $\mathbb{E}^{\mu}_I(g) \equiv \frac{1}{\mu(I)}\int_I g\mu dx$. A weight $\mu \in A_p$ if $\sup_I \langle \mu \rangle_I \left\langle \mu^{p'-1} \right\rangle_I^{p-1} \equiv [\mu]_{A_p} < \infty$.

In [2] Bloom considers the behavior of the commutator

$$[b, H] : L^p(\lambda) \mapsto L^p(\mu)$$

where H is the Hilbert transform. When the weights $\mu = \lambda \in A_2$ then it is well known that this boundedness of this commutator is characterized by the membership of $b \in BMO$, [1]. Bloom obtains an equivalent characterization of the commutators in his setting, thought the BMO he works with is particular to the weights at hand. Let $\rho = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$, and then define:

$$\|b\|_{BMO_\rho} \equiv \sup_I \left(\frac{1}{\rho(I)} \int_I |b(x) - \langle b \rangle_I|^2 \, dx \right)^{\frac{1}{2}}.$$

Bloom then proves the following:

Theorem 1 (Bloom, [2, Theorem 4.2])**.** *Let $1 < p < \infty$, $\mu, \lambda \in A_p$. Set $\rho = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$. Then,*

$$\left\| [b, H] : L^2(\mu) \to L^2(\lambda) \right\| \approx \|b\|_{BMO_\rho}.$$

This talk gave an alternate proof of Theorem 1 in the case when $p = 2$. We focus on this case since it contains the key ideas for the more general result. In a forthcoming paper the authors will show how Bloom's result can be extended to all Calderón-Zygmund operators in arbitrary dimension and when $1 < p < \infty$.

2. Two Weight Inequalities for Paraproduct Operators

Let $\mathcal{D}$ denote the standard dyadic lattice on $\mathbb{R}$. And, let h_I denote the Haar function adapted to the interval $I \in \mathcal{D}$. The 'paraproduct' operator with symbol function b and its adjoint are defined by

$$\Pi_b \equiv \sum_{I\in\mathcal{D}} \widehat{b}(I) h_I \otimes \frac{\mathbf{1}_I}{|I|} \quad \Pi_b^* \equiv \sum_{I\in\mathcal{D}} \widehat{b}(I) \frac{\mathbf{1}_I}{|I|} \otimes h_I .$$

Our first result provides necessary and sufficient conditions so that Π_b and Π_b^* are bounded between weighted spaces $L^2(\mathbb{R};\mu)$ and $L^2(\mathbb{R};\lambda)$. For weights $\mu, \lambda \in A_2$, define

$$\mathbf{B}_2[\mu,\lambda] \equiv \sup_{K\in\mathcal{D}} \mu^{-1}(K)^{-1/2} \left\| \sum_{I\,:\,I\subset K} \widehat{b}(I) \langle \mu^{-1}\rangle_I h_I \right\|_{L^2(\lambda)} \tag{7}$$

Theorem 2. *Let $\mu, \lambda \in A_2$. Suppose that $\mathbf{B}_2[\mu,\lambda]$ and $\mathbf{B}_2[\lambda^{-1},\mu^{-1}]$ finite. Then we have*

$$\left\|\Pi_b : L^2(\mu) \to L^2(\lambda)\right\| \lesssim [\lambda]_{A_2} \mathbf{B}_2[\mu,\lambda] \tag{8}$$

$$\left\|\Pi_b^* : L^2(\mu) \to L^2(\lambda)\right\| \lesssim [\mu]_{A_2} \mathbf{B}_2[\lambda^{-1},\mu^{-1}]. \tag{9}$$

Conversely, we have that

$$\left\|\Pi_b : L^2(\mu) \to L^2(\lambda)\right\| \gtrsim \mathbf{B}_2[\mu,\lambda] \tag{10}$$

$$\left\|\Pi_b^* : L^2(\mu) \to L^2(\lambda)\right\| \gtrsim \mathbf{B}_2[\lambda^{-1},\mu^{-1}]. \tag{11}$$

Using this Theorem we have the following result, providing more information and a new proof of Bloom's result.

Theorem 3. *Let $\mu, \lambda \in A_2$. Suppose that $\mathbf{B}_2[\mu,\lambda]$ and $\mathbf{B}_2[\lambda^{-1},\mu^{-1}]$ are finite. Then $[b,H] : L^2(\mu) \to L^2(\lambda)$ with $\left\|[b,H] : L^2(\mu) \to L^2(\lambda)\right\| \lesssim$*

$$[\lambda]_{A_2}[\mu]_{A_2}(\mathbf{B}_2[\mu,\lambda] + \mathbf{B}_2[\lambda^{-1},\mu^{-1}]) + [\lambda]_{A_2}^2 \mathbf{B}_2[\mu,\lambda] + [\mu]_{A_2}^2 \mathbf{B}_2[\lambda^{-1},\mu^{-1}].$$

Conversely, if $[b,H] : L^2(\mu) \to L^2(\lambda)$ is bounded, then $\mathbf{B}_2[\mu,\lambda]$ and $\mathbf{B}_2[\lambda^{-1},\mu^{-1}]$ are finite. In particular, up to the A_2 characteristic of μ and λ, we have that:

$$\mathbf{B}_2[\mu,\lambda] + \mathbf{B}_2[\lambda^{-1},\mu^{-1}] \approx \left\|[b,H] : L^2(\mu) \to L^2(\lambda)\right\| .$$

The proof of Theorem 3 uses Petermichl's dyadic shifts to represent the Hilbert transform, [3]. The commutator $[b,H]$ can be expressed in terms of the paraproducts Π_b, Π_b^*, the dyadic shifts and an "error" term that is of Haar multiplier type. These are then easily controlled by (8) and (9).

References

[1] Oleksandra V. Beznosova, *Linear bound for the dyadic paraproduct on weighted Lebesgue space $L_2(w)$*, J. Funct. Anal. **255** (2008), no. 4, 994–1007.

[2] S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 103–122.

[3] S. Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 6, 455–460.

Toeplitz algebra of Toeplitz space?

Jingbo Xia, SUNY Buffalo

Let $L_a^2(\mathbf{B}, dv)$ be the standard Bergman space on the unit ball $\mathbf{B}$ in $\mathbf{C}^n$. Recall that the Toeplitz operator T_f on $L_a^2(\mathbf{B}, dv)$ is defined by the formula

$$T_f h = P(fh), \quad h \in L_a^2(\mathbf{B}, dv),$$

where P is the orthogonal projection from $L^2(\mathbf{B}, dv)$ to $L_a^2(\mathbf{B}, dv)$. Also recall that the *Toeplitz algebra* $\mathcal{T}$ on $L_a^2(\mathbf{B}, dv)$ is the C^*-algebra generated by the collection of Toeplitz operators $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$.

It was first discovered in [9] that *localization* is a powerful tool for analyzing operators on reproducing-kernel Hilbert spaces. Recently, this idea was further explored in [3], where Isralowitz, Mitkovski and Wick introduced the notion of *weakly localized operators* on the Bergman space. Localization on the Bergman space involves the Bergman metric

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B}.$$

For $z \in \mathbf{B}$ and $r > 0$, denote $D(z, r) = \{w \in \mathbf{B} : \beta(z, w) < r\}$. Recall that the formula

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}$$

defines the standard Möbius-invariant measure on $\mathbf{B}$. Write k_z, $z \in \mathbf{B}$, for the normalized reproducing kernel for the Bergman space. Below we give a slightly more refined version of localization due to Isralowitz, Mitkovski and Wick. Our refinement lies in the realization that we can define a class of localized operators for each given localization parameter s.

Definition 1.1. Let a positive number $(n-1)/(n+1) < s < 1$ be given.
(a) A bounded operator B on the Bergman space $L_a^2(\mathbf{B}, dv)$ is said to be s-weakly localized if it satisfies the conditions

$$\sup_{z \in \mathbf{B}} \int |\langle Bk_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\sup_{z \in \mathbf{B}} \int |\langle B^* k_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\lim_{r \to \infty} \sup_{z \in \mathbf{B}} \int_{\mathbf{B} \setminus D(z,r)} |\langle Bk_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) = 0 \quad \text{and}$$

$$\lim_{r \to \infty} \sup_{z \in \mathbf{B}} \int_{\mathbf{B} \setminus D(z,r)} |\langle B^* k_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) = 0.$$

(b) Let $\mathcal{A}_s$ denote the collection of s-weakly localized operators defined as above.
(c) Let $C^*(\mathcal{A}_s)$ denote the C^*-algebra generated by $\mathcal{A}_s$.

For each $(n-1)/(n+1) < s < 1$, the simplest examples of s-weakly localized operators are the Toeplitz operators with bounded symbols. Indeed it was shown in [3] that each $\mathcal{A}_s$ is a $*$-algebra that contains $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$. Hence $C^*(\mathcal{A}_s) \supset \mathcal{T}$.

In [8], Suárez showed that for $A \in \mathcal{T}$, the condition

$$\lim_{|z|\uparrow 1} \langle Ak_z, k_z\rangle = 0$$

implies that A is compact. In [3], Isralowitz, Mitkovski and Wick showed that for $A \in C^*(\mathcal{A}_s)$, the above limit also implies that A is compact. Moreover, the introduction of the notion of weakly localized operators in [3] has the added virtue that it significantly simplifies the work necessary to obtain this result. This motivated us to carefully examine the inclusion relation

$$\mathcal{T} \subset C^*(\mathcal{A}_s).$$

The Toeplitz algebra $\mathcal{T}$ is certainly considered to be much better understood than $C^*(\mathcal{A}_s)$. It is known, for example, that $\mathcal{T}$ coincides with its commutator ideal [6,4]. Thus an obvious question is, is the C^*-algebra $C^*(\mathcal{A}_s)$ structurally different from $\mathcal{T}$? In fact, one may raise the even more basic

Question 1.2. Is the inclusion $\mathcal{T} \subset C^*(\mathcal{A}_s)$ proper for any $(n-1)/(n+1) < s < 1$? Is there any difference between $C^*(\mathcal{A}_s)$ and $C^*(\mathcal{A}_t)$ for $s \neq t$ in the interval $((n-1)/(n+1), 1)$?

The answer, as it turns out, is somewhat surprising:

Theorem 1.3. *For every $(n-1)/(n+1) < s < 1$ we have $C^*(\mathcal{A}_s) = \mathcal{T}$.*

An immediate consequence of Theorem 1.3 is, of course, that $C^*(\mathcal{A}_s) = C^*(\mathcal{A}_t)$ for all $s, t \in ((n-1)/(n+1), 1)$. We emphasize that this equality at the level of C^*-algebras is obtained without knowing whether there is any kind of inclusion relation between the classes $\mathcal{A}_s$ and $\mathcal{A}_t$ in the case $s \neq t$.

Although Question 1.2 was the original motivation for this investigation, our approach to this problem naturally leads us to a stronger result, a result that simultaneously answers a more basic question in the theory of Toeplitz operators. Let us introduce

Definition 1.4. Let $\mathcal{T}^{(1)}$ denote the closure of $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ with respect to the operator norm.

Below is our main result, which not only answers Question 1.2, but also tells us something significant about the Toeplitz algebra $\mathcal{T}$ itself.

Theorem 1.5. *For every $(n-1)/(n+1) < s < 1$ we have $\mathcal{T}^{(1)} = C^*(\mathcal{A}_s)$. Consequently, $\mathcal{T}^{(1)} = \mathcal{T} = C^*(\mathcal{A}_s)$.*

The documented history of interest in $\mathcal{T}^{(1)}$ can be traced at least back to [1,2], where Engliš showed that it contains all the compact operators on $L^2_a(\mathbf{B}, dv)$. In retrospect, this was really a hint at the things to come.

Later in [7], Suárez took another look at $\mathcal{T}^{(1)}$. There he introduced a sequence of higher Berezin transforms $B_1, \ldots, B_k, \ldots$, which are generalizations of the original Berezin transform B_0. At the end of the paper, Suárez expressed his belief that every operator S in $\mathcal{T}$ is the limit in operator norm of the sequence of Toeplitz operators $\{T_{B_k(S)}\}$. If this is true, then it certainly implies that $\mathcal{T}^{(1)} = \mathcal{T}$. One can only speculate that, perhaps, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ was what Suárez had in mind all along, and the higher Berezin transforms were his tools to try to prove it. While we still do not know if it is true that

$$\lim_{k\to\infty} \|T_{B_k(S)} - S\| = 0$$

for every $S \in \mathcal{T}$, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ is now proven using completely different ideas. From the proof of Theorem 1.5, the reader will see that the approximation of a general $S \in \mathcal{T}$ by Toeplitz operators is quite complicated: it takes several stages.

Let us give an outline for the proof of Theorem 1.5. First of all, we need the notion of *separated* sets in $\mathbf{B}$. A set $\Gamma \subset \mathbf{B}$ is said to be separated if there is a $\delta = \delta(\Gamma) > 0$ such that $\beta(u, v) \geq \delta$ for all $u \neq v$ in Γ. The key technical estimate for the proof is the following:

Lemma 2.6. *Given any separated set* Γ *in* $\mathbf{B}$, *there exists a constant* $0 < B(\Gamma) < \infty$ *such that the following estimate holds*: *Let* $\{h_u : u \in \Gamma\}$ *be functions in* $H^\infty(\mathbf{B})$ *such that* $\sup_{u\in\Gamma} \|h_u\|_\infty < \infty$, *and let* $\{e_u : u \in \Gamma\}$ *be any orthonormal set. Then*

$$\left\| \sum_{u\in\Gamma} (U_u h_u) \otimes e_u \right\| \leq B(\Gamma) \sup_{u\in\Gamma} \|h_u\|_\infty.$$

Since each $\mathcal{A}_s$ is known to be a $*$-algebra that contains $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ [3], it suffices to show that $\mathcal{A}_s \subset \mathcal{T}^{(1)}$. An elementary C^*-algebraic argument further reduces this to the proof of the inclusion

$$T_\Phi \mathcal{A}_s T_\Phi \subset \mathcal{T}^{(1)}$$

for a suitably chosen Toeplitz operator T_Φ that is both positive and invertible. We can pick the function Φ in such a way that for every $B \in \mathcal{A}_s$, the operator $T_\Phi B T_\Phi$ is “resolved” in the form

$$T_\Phi B T_\Phi = \iint_{D(0,2)\times D(0,2)} E_w B E_z d\lambda(w) d\lambda(z),$$

where each E_z is a sum of rank-one operators over a lattice:

$$E_z = \sum_{u\in\mathcal{L}} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}.$$

A consequence of Lemma 2.6 is that the map $(w, z) \mapsto E_w B E_z$ is continuous with respect to the operator norm. This norm continuity immediately implies that $T_\Phi B T_\Phi$ is contained in the norm closure of the linear span of

$$\{E_w B E_z : w, z \in \mathbf{B}\}.$$

Thus we can complete the proof by showing that $E_w B E_z \in \mathcal{T}^{(1)}$ for all $z, w \in \mathbf{B}$. One can think of $E_w B E_z$ as an infinite matrix. The localization condition for B ensures that the terms in $E_w B E_z$ that are “far from the diagonal” form an operator of small norm. The rest of the terms in $E_w B E_z$ are a linear combination of operators in a special class $\mathcal{D}_0$ (see Definition 3.1 below). In other words, $E_w B E_z$ can be approximated in norm by operators in the linear span of $\mathcal{D}_0$. Then, with several applications of the estimate in Lemma 2.6, we are able to show that $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$, accomplishing our goal.

Definition 3.1. Let $\mathcal{D}_0$ denote the collection of operators of the form

$$\sum_{u\in\Gamma} c_u k_u \otimes k_{\gamma(u)},$$

where Γ is any separated set in $\mathbf{B}$, $\{c_u : u \in \Gamma\}$ is any bounded set of complex coefficients, and $\gamma : \Gamma \to \mathbf{B}$ is any map for which there is a $0 < C < \infty$ such that

$$\beta(u, \gamma(u)) \leq C$$

for every $u \in \Gamma$.

References

1. M. Engliš, Toeplitz operators on Bergman-type spaces, Ph.D. Dissertation, Prague 1991.
2. M. Engliš, Density of algebras generated by Toeplitz operator on Bergman spaces, Ark. Mat. **30** (1992), 227-243.
3. J. Isralowitz, M. Mitkovski and B. Wick, Localization and compactness in Bergman and Fock spaces, to appear in Indiana Univ. Math. J., arXiv:1306.0316
4. T. Le, On the commutator ideal of the Toeplitz algebra on the Bergman space of the unit ball in $\mathbf{C}^n$, J. Operator Theory **60** (2008), 149-163.
5. K. Nam, D. Zheng and C. Zhong, m-Berezin transform and compact operators, Rev. Mat. Iberoam. **22** (2006), 867-892.
6. D. Suárez, The Toeplitz algebra on the Bergman space coincides with its commutator ideal, J. Operator Theory **51** (2004), 105-114.
7. D. Suárez, Approximation and the n-Berezin transform of operators on the Bergman space, J. Reine Angew. Math. **581** (2005), 175-192.
8. D. Suárez, The essential norm of operators in the Toeplitz algebra on $A^p(\mathbf{B}_n)$, Indiana Univ. Math. J. **56** (2007), 2185-2232.
9. J. Xia and D. Zheng, Localization and Berezin transform on the Fock space, J. Funct. Anal. **264** (2013), 97-117.

Curvature properties of Cowen-Douglas classes and of Forelli-Rudin constructions

Genkai Zhang, Chalmers University

In the present work we shall study the positivity of the curvature of Cowen-Douglas operators and its infinite-dimensional analogue, the Forelli-Rudin construction of Bergman spaces.

Let Ω be an open bounded domain in $\mathbb{C}^m$ and let $T = (T_1, T_2, \cdots, T_m)$ be a Cowen-Douglas [2] tuple of commuting operators in $B_n(\Omega)$ on a Hilbert space H. The eigenspaces form a $\mathbb{C}^n$-dimensional vector bundle over Ω,

$$E_T \subset \Omega \times H, H \supset E_w := Ker(T - w) \to w \in \Omega,$$

with the Hermitian metric given by H. Let $R(X, Y) = R(X, Y)(w) : E_w \mapsto E_w$ be the curvature of the vector bundle,

$$R(X, Y)u = [\mathcal{D}_X, \bar{\partial}_Y]u = -\bar{\partial}_Y \mathcal{D}_X u$$

when acting on holomorphic sections near w, where $\mathcal{D} + \bar{\partial}$ is the Chern connection.

We prove the following result

Theorem 1. *(1) The curvature for Cowen-Douglas operator T is Nagano seminegative and Griffith negative. In particular the Ricci curvature is always negative.*

(2) Let (H, K) be a Hilbert space of vector space V-valued holomorphic functions on a domain Ω. Suppose that all the constant functions are in H. Then the curvature is Nagano negative.

(3) Consider the infinite flag bundle

$$H \supset \cdots Ker(T-w)^2 \supset Ker(T-w) \mapsto w \in \Omega$$

Let π_n be the projections onto the complementary subspaces

$$0 \le \cdots \le \pi_2 \le \pi_1 \le I.$$

Then the curvature $R^n(X, X)$ of the bundle $Ker(T-w)^n \to w \in \Omega$ is given by

$$(R^n(X,X)u,u)(w_0) = -\sum_{j=1}^{\infty} \|\pi_{n+j}\mathcal{D}_X^{n+j}u\|^2(w_0).$$

Here $u = u(w)$ is a local holomorphic section of the bundle.

We recall that the curvature R of a Hermitian vector bundle (E, g) with metric g is called Griffith positive if for any nonzero tangent vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$ in the fiber space,

$$g(R(u,\bar{u})(v),v) = \sum R_{i\bar{j}\alpha\bar{\beta}} u^i \bar{u}^j v^\alpha \bar{v}^\beta > 0;$$

it is called Nakano positive if for any nonzero vector $u = u^{i\alpha}\frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0,$$

i.e., the associated Hermitian form $u \otimes v \mapsto g(R(u,u)v,v)$ extends to a positive definite Hermitian form on the total space $T^{(1,0)} \otimes E$.

Remark 1. This generalizes recent results of Biswas-Keshari-Misra [4].

An infinite-dimensional analogue of the Cowen-Douglas bundle is the Forelli-Rudin construction, or more generally, the bundle of Bergman spaces for a fibration of a complex domain. We apply our earlier results [3] to the formula recently found by Berndtsson [1] for the curvature of the bundle and compute the Dixmier trace for the curvature operator.

Theorem 2. *Let $\mathcal{X}$ be a pseudo-convex domain fibered over Ω, $\mathcal{X}_t \mapsto t$, $\mathcal{X}$, Ω, and $\mathcal{X}_t$ being pseudo-convex. Let ϕ be a plurisubharmonic function on Ω, smooth on the closure, and ϕ^t its restriction on $\mathcal{X}_t$. Assume that each $\mathcal{X}_t$ is of dimension $n \ge 2$ and is strongly pseudo-convex with smooth boundary. Consider the bundle of Bergman spaces $L^2_a(\mathcal{X}_t, e^{-\phi^t}) \mapsto t \in \Omega$. Then the operator*

$$R(X,X) - T_{\partial_X \partial_{\bar{X}}\phi}, \quad X \in T_w^{(1,0)}(\Omega),$$

is of Dixmier class $\mathcal{L}^{n,\infty}$ and

$$Tr_\omega(R(X,X) - T_{\partial_X \partial_{\bar{X}}\phi})^n = -\int_{\partial_X \phi} J(\partial_X \phi),$$

where $J(\psi)$ is an explicit functional written in terms of the boundary CR-Poisson bracket.T

Here T_ψ in the Toeplitz operator on the Bergman space $L^2_a(\mathcal{X}_t, e^{-\phi^t})$.

References

[1] B. Berndtsson, "Curvature of vector bundles associated to holomorphic fibrations." Annals of mathematics (2009): 531-560.

[2] M. Cowen, and R. Douglas. "Complex geometry and operator theory." Acta Mathematica 141.1 (1978): 187-261.

[3] M. Engliš, M., and G. Zhang, "Hankel operators and the Dixmier trace on strictly pseudoconvex domains." Documenta Mathematica 15 (2010): 601-622.

[4] Shibananda Biswas, Dinesh Kumar Keshari, and Gadadhar Misra, "Infinitely divisible metrics and curvature inequalities for operators in the Cowen-Douglas class", arXiv:1207.2025