

Toeplitz algebra or Toeplitz linear space?

Jingbo Xia

SUNY Buffalo, NY

Basic setting and notation:

$\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$, the unit ball in \mathbf{C}^n .

dv = the volume measure on \mathbf{B} with the normalization $v(\mathbf{B}) = 1$.

$L^2_a(\mathbf{B}, dv)$: the Bergman space on \mathbf{B} , which is the collection of analytic functions in $L^2(\mathbf{B}, dv)$.

$P : L^2(\mathbf{B}, dv) \rightarrow L^2_a(\mathbf{B}, dv)$: the orthogonal projection.

Toeplitz operator : $T_f h = P(fh)$, $h \in L^2_a(\mathbf{B}, dv)$.

\mathcal{T} : the Toeplitz algebra, i.e., the C^* -algebra generated by the collection of Toeplitz operators $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$ with **bounded** symbols.

Normalized reproducing kernel for the Bergman space:

$$k_z(w) = \frac{(1 - |z|^2)^{(n+1)/2}}{(1 - \langle w, z \rangle)^{n+1}}, \quad z, w \in \mathbf{B}.$$

Theorem. (Suárez, 2007) For $A \in \mathcal{T}$, if

$$\lim_{|z| \uparrow 1} \langle Ak_z, k_z \rangle = 0,$$

then A is a compact operator.

This was a drastic improvement of an earlier result by Axler and Zheng.

In 2012, Bauer and Isralowitz proved the **Fock space** analogue of Suárez's result. This led Zheng and myself to take a look at what was going on.

Fock space : $H^2(\mathbf{C}^n, d\mu)$, where $d\mu$ is the Gaussian measure

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z).$$

Normalized reproducing kernel for the Fock space:

$$k_z^{\text{Fock}}(\zeta) = e^{-|z|^2} e^{\langle \zeta, z \rangle}, \quad z, \zeta \in \mathbf{C}^n.$$

Definition. (X. and Zheng, 2013) A bounded operator B on $H^2(\mathbf{C}^n, d\mu)$ is said to be **sufficiently localized** if there exist constants $2n < \beta < \infty$ and $0 < C < \infty$ such that

$$|\langle Bk_z^{\text{Fock}}, k_w^{\text{Fock}} \rangle| \leq \frac{C}{(1 + |z - w|)^\beta}$$

for all $z, w \in \mathbf{C}^n$.

It is easy to show that every Toeplitz operator **with a bounded symbol** on $H^2(\mathbf{C}^n, d\mu)$ is sufficiently localized.

Theorem. (X. and Zheng, 2013) Let \mathcal{SL} denote the collection of sufficiently localized operators on $H^2(\mathbf{C}^n, d\mu)$, and let $C^*(\mathcal{SL})$ be the C^* -algebra generated by \mathcal{SL} . For $A \in C^*(\mathcal{SL})$, if

$$\lim_{|z| \rightarrow \infty} \langle Ak_z^{\text{Fock}}, k_z^{\text{Fock}} \rangle = 0,$$

then A is a compact operator.

The point is this: the use of the notion of **localization** greatly simplifies the work in the proof.

This led Isralowitz, Mitkovski and Wick to introduce the notion of **localization** for operators on the **Bergman space**, which involves the Bergman metric.

Möbius transform:

$$\varphi_u(\zeta) = \frac{1}{1 - \langle \zeta, u \rangle} \left\{ u - \frac{\langle \zeta, u \rangle}{|u|^2} u - (1 - |u|^2)^{1/2} \left(\zeta - \frac{\langle \zeta, u \rangle}{|u|^2} u \right) \right\}$$

if $u \in \mathbf{B} \setminus \{0\}$. $\varphi_0(\zeta) = -\zeta$.

Möbius invariant measure on the ball:

$$d\lambda(\zeta) = \frac{dv(\zeta)}{(1 - |\zeta|^2)^{n+1}}.$$

Bergman metric:

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbf{B}.$$

Ball in Bergman metric: $D(z, r) = \{w \in \mathbf{B} : \beta(z, w) < r\}$.

The following is a slightly refined version of the notion of localization introduced by Isralowitz, Mitkovski and Wick:

Definition 1.1. (2014) Let $(n-1)/(n+1) < s < 1$.

(a) A bounded operator B on the Bergman space $L_a^2(\mathbf{B}, d\nu)$ is said to be **s-weakly localized** if it satisfies the conditions

$$\sup_{z \in \mathbf{B}} \int |\langle Bk_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\sup_{z \in \mathbf{B}} \int |\langle B^*k_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\lim_{r \rightarrow \infty} \sup_{z \in \mathbf{B}} \int_{\mathbf{B} \setminus D(z,r)} |\langle Bk_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) = 0 \quad \text{and}$$

$$\lim_{r \rightarrow \infty} \sup_{z \in \mathbf{B}} \int_{\mathbf{B} \setminus D(z,r)} |\langle B^*k_z, k_w \rangle| \left(\frac{1 - |w|^2}{1 - |z|^2} \right)^{s(n+1)/2} d\lambda(w) = 0.$$

(b) Let \mathcal{A}_s denote the collection of s -weakly localized operators defined as above.

(c) Let $C^*(\mathcal{A}_s)$ denote the C^* -algebra generated by \mathcal{A}_s .

Theorem. (Isralowitz, Mitkovski and Wick, preprint 2014)
For $A \in C^*(\mathcal{A}_s)$, $(n-1)/(n+1) < s < 1$, if

$$\lim_{|z| \uparrow 1} \langle Ak_z, k_z \rangle = 0,$$

then A is a compact operator.

They also showed that each \mathcal{A}_s is a $*$ -algebra that contains the Toeplitz operators $\{T_f : f \in L^\infty(\mathbf{B}, dv)\}$. Therefore $C^*(\mathcal{A}_s) \supset \mathcal{T}$, and consequently their result generalizes Suárez's result of 2007.

Late last summer, I decided to take a closer look at the inclusion relation $C^*(\mathcal{A}_s) \supset \mathcal{T}$, partly because the Toeplitz algebra \mathcal{T} is a much better understood object than $C^*(\mathcal{A}_s)$, or at least so we think.

For example, \mathcal{T} is known to coincide with its commutator ideal (Suárez 2004 and Le 2008). Is the same true for $C^*(\mathcal{A}_s)$?

In fact, one may raise the even more basic

Question 1.2. Is the inclusion $C^*(\mathcal{A}_s) \supset \mathcal{T}$ proper for any $(n-1)/(n+1) < s < 1$? Is there any difference between $C^*(\mathcal{A}_s)$ and $C^*(\mathcal{A}_t)$ for $s \neq t$ in the interval $((n-1)/(n+1), 1)$?

The answer, as it turns out, is somewhat surprising:

Theorem 1.3. *For every $(n - 1)/(n + 1) < s < 1$ we have $C^*(\mathcal{A}_s) = \mathcal{T}$.*

An immediate consequence of Theorem 1.3 is, of course, that $C^*(\mathcal{A}_s) = C^*(\mathcal{A}_t)$ for all $s, t \in ((n - 1)/(n + 1), 1)$. I want to emphasize that this equality at the level of C^* -algebras is obtained without knowing whether there is any kind of inclusion relation between the classes \mathcal{A}_s and \mathcal{A}_t in the case $s \neq t$.

Although Question 1.2 was the original motivation for this investigation, the solution of this problem naturally leads to a stronger result, a result that simultaneously settles a much older question. Let us introduce

Definition 1.4. Let $\mathcal{T}^{(1)}$ denote the closure of

$$\{T_f : f \in L^\infty(\mathbf{B}, d\nu)\}$$

with respect to the operator norm.

Below is the main result of the talk, which not only answers Question 1.2, but also tells us something about the Toeplitz algebra \mathcal{T} that was not previously known.

Theorem 1.5. *For every $(n - 1)/(n + 1) < s < 1$ we have $\mathcal{T}^{(1)} = C^*(\mathcal{A}_s)$. Consequently, $\mathcal{T}^{(1)} = \mathcal{T} = C^*(\mathcal{A}_s)$.*

The documented history of interest in $\mathcal{T}^{(1)}$ can be traced at least back to Engliš's PhD dissertation in 1991, where he showed that it contains all the compact operators on $L_a^2(\mathbf{B}, dv)$. In retrospect, this was really a hint at the things to come.

Later in 2005, Suárez took another look at $\mathcal{T}^{(1)}$. He introduced a sequence of higher Berezin transforms B_1, \dots, B_k, \dots , which are generalizations of the original Berezin transform B_0 . At the end of his 2005 paper, Suárez expressed his belief that every operator S in \mathcal{T} is the limit in operator norm of the sequence of Toeplitz operators $\{T_{B_k(S)}\}$. If this is true, then it certainly implies that $\mathcal{T}^{(1)} = \mathcal{T}$.

One can only speculate that, perhaps, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ was what Suárez had in mind all along, and the higher Berezin transforms were his tools to try to prove it. While we still do not know if it is true that

$$\lim_{k \rightarrow \infty} \|T_{B_k(S)} - S\| = 0$$

for every $S \in \mathcal{T}$, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ is now proven using completely different ideas. The proof of Theorem 1.5 shows that the approximation of a general $S \in \mathcal{T}$ by Toeplitz operators is actually quite complicated: **it takes several stages.**

So how do we prove Theorem 1.5? Since we know that each \mathcal{A}_S is a $*$ -algebra that contains $\{T_f : f \in L^\infty(\mathbf{B}, d\nu)\}$, it suffices to show

$$\mathcal{A}_S \subset \mathcal{T}^{(1)}.$$

Observation. If there is a function Φ satisfying the condition

$$(**) \quad 0 < c \leq \Phi \leq C < \infty \quad \text{on } \mathbf{B}$$

such that $T_\Phi \mathcal{A}_S T_\Phi \subset \mathcal{T}^{(1)}$, then $\mathcal{A}_S \subset \mathcal{T}^{(1)}$.

Proof. Let $B \in \mathcal{A}_s$. Since \mathcal{A}_s is known to be an algebra that contains all the Toeplitz operators with bounded symbol, we have $T_\Phi^j B T_\Phi^k \in \mathcal{A}_s$ for all $j, k \in \mathbf{Z}_+$. By assumption, this means

$$T_\Phi^{j+1} B T_\Phi^{k+1} \in \mathcal{T}^{(1)} \quad \text{for all } j, k \in \mathbf{Z}_+.$$

Let $C^*(T_\Phi)$ be the unital C^* -algebra generated by T_Φ . Since T_Φ is self-adjoint, the above implies

$$X T_\Phi B X T_\Phi \in \mathcal{T}^{(1)} \quad \text{for every } X \in C^*(T_\Phi).$$

Condition (**) guarantees that T_Φ is invertible. By elementary C^* -algebra, the inverse T_Φ^{-1} , whenever it exists, has to be in $C^*(T_\Phi)$. Thus, taking $X = T_\Phi^{-1}$ in the above, we have $B \in \mathcal{T}^{(1)}$.

□

Definition 2.1. A subset Γ of \mathbf{B} is said to be **separated** if there is a $\delta = \delta(\Gamma) > 0$ such that the inequality $\beta(u, v) \geq \delta$ holds for all $u \neq v$ in Γ .

Let \mathcal{L} be a subset of \mathbf{B} which is **maximal** with respect to the property that

$$D(u, 1) \cap D(v, 1) = \emptyset \quad \text{for all } u \neq v \text{ in } \mathcal{L}.$$

This \mathcal{L} is a separated set. The maximality of \mathcal{L} implies that

$$\bigcup_{u \in \mathcal{L}} D(u, 2) = \mathbf{B}.$$

Define the function

$$\Phi = \sum_{u \in \mathcal{L}} \chi_{D(u,2)}.$$

By the separatedness of \mathcal{L} and the covering property of $\{D(u, 2) : u \in \mathcal{L}\}$, we have

$$1 \leq \Phi \leq N \quad \text{on } \mathbf{B}$$

for some natural number N . By the **Observation**, it suffices to prove $T_\Phi \mathcal{A}_s T_\Phi \subset \mathcal{T}^{(1)}$ for this particular Φ .

But for this particular Φ , the Toeplitz operator T_Φ has a nice representation.

For any $f \in L^\infty(\mathbf{B}, d\nu)$, the Toeplitz operator T_f has the **integral representation**

$$T_f = \int f(w) k_w \otimes k_w d\lambda(w).$$

For the Φ just defined above, by the Möbius invariance of β and $d\lambda$, we have

$$T_\Phi = \int \Phi(w) k_w \otimes k_w d\lambda(w) = \int_{D(0,2)} E_z d\lambda(z),$$

where

$$E_z = \sum_{u \in \mathcal{L}} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)},$$

a sum of rank-one operators over the lattice \mathcal{L} . The correct way to read this is that the set

$$\{\varphi_u(z) : u \in \mathcal{L}\}$$

is a “translation” of the lattice \mathcal{L} by z . Equivalently, one thinks of $\varphi_u(z)$ as a perturbation of u by z , **not the other way around**.

For each $z \in \mathbf{B}$, the formula

$$(U_z h)(\zeta) = h(\varphi_z(\zeta))k_z(\zeta), \quad h \in L_a^2(\mathbf{B}; dv),$$

defines a unitary operator on the Bergman space.

Key estimate:

Lemma 2.6. *Given any separated set Γ in \mathbf{B} , there exists a constant $0 < B(\Gamma) < \infty$ such that the following estimate holds: Let $\{h_u : u \in \Gamma\}$ be functions in $H^\infty(\mathbf{B})$ such that $\sup_{u \in \Gamma} \|h_u\|_\infty < \infty$, and let $\{e_u : u \in \Gamma\}$ be any orthonormal set. Then*

$$\left\| \sum_{u \in \Gamma} (U_u h_u) \otimes e_u \right\| \leq B(\Gamma) \sup_{u \in \Gamma} \|h_u\|_\infty.$$

Let $B \in \mathcal{A}_s$ be given. By the integral representation of T_ϕ , we can “resolve” $T_\phi B T_\phi$ in the form

$$T_\phi B T_\phi = \iint_{D(0,2) \times D(0,2)} E_w B E_z d\lambda(w) d\lambda(z).$$

Lemma 2.6 has a number of implications, and one of the implications is that the map

$$(w, z) \mapsto E_w B E_z$$

is **continuous with respect to the operator norm**. This norm continuity immediately implies that $T_\phi B T_\phi$ is contained in the norm closure of the linear span of

$$\{E_w B E_z : w, z \in \mathbf{B}\}.$$

Thus we can complete the proof by showing that $E_w B E_z \in \mathcal{T}^{(1)}$ for all $z, w \in \mathbf{B}$.

One can think of $E_w B E_z$ as an infinite matrix. The localization condition for B ensures that the terms in $E_w B E_z$ that are “far from the diagonal” form an operator of small norm. The rest of the terms in $E_w B E_z$ are a linear combination of operators in a special class \mathcal{D}_0 (see below). In other words, $E_w B E_z$ can be approximated in norm by operators in the linear span of \mathcal{D}_0 . Then, with several applications of the estimate in Lemma 2.6, we are able to show that $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$, completing the proof of Theorem 1.5.

Let me say a little more about \mathcal{D}_0 and about how the inclusion $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$ is proved, which is technically the most interesting part.

Definition 3.1. Let \mathcal{D}_0 denote the collection of operators of the form

$$\sum_{u \in \Gamma} c_u k_u \otimes k_{\gamma(u)},$$

where Γ is any separated set in \mathbf{B} , $\{c_u : u \in \Gamma\}$ is any bounded set of complex coefficients, and $\gamma : \Gamma \rightarrow \mathbf{B}$ is any map for which there is a $0 < C < \infty$ such that

$$\beta(u, \gamma(u)) \leq C$$

for every $u \in \Gamma$.

Note that the condition $\beta(u, \gamma(u)) \leq C$ is in fact a form of **localization condition**. That is, the operators in \mathcal{D}_0 are all localized in this particular way.

The proof of the inclusion $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$ takes the following steps:

Proposition 3.5. *Suppose that Γ is a separated set in \mathbf{B} . Furthermore, suppose that $\{c_u : u \in \Gamma\}$ are complex numbers satisfying the condition*

$$\sup_{u \in \Gamma} |c_u| < \infty.$$

Then for each $z \in \mathbf{B}$, the operator

$$Y_z = \sum_{u \in \Gamma} c_u k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}$$

belongs to $\mathcal{T}^{(1)}$.

Proof. It follows from Lemma 2.6 that the map $z \mapsto Y_z$ is continuous in operator norm. Therefore

$$\lim_{\epsilon \downarrow 0} \|Y_0 - A_\epsilon\| = 0, \quad \text{where } A_\epsilon = \frac{1}{\lambda(D(0, \epsilon))} \int_{D(0, \epsilon)} Y_z d\lambda(z).$$

By the Möbius invariance of β and $d\lambda$, we have $A_\epsilon = T_{f_\epsilon}$, where

$$f_\epsilon = \frac{1}{\lambda(D(0, \epsilon))} \sum_{u \in \Gamma} c_u \chi_{D(u, \epsilon)},$$

which is a bounded function for sufficiently small $\epsilon > 0$. Thus $Y_0 \in \mathcal{T}^{(1)}$, proving the proposition for the case $z = 0$. For a general $z \in \mathbf{B}$, it now suffices to observe that the set

$$\{\varphi_u(z) : u \in \Gamma\}$$

is the union of a finite number of separated sets in \mathbf{B} . \square

Unnormalized reproducing kernel:

$$K_z(\zeta) = \frac{1}{(1 - \langle \zeta, z \rangle)^{n+1}}, \quad z, \zeta \in \mathbf{B}.$$

For each pair of $\alpha \in \mathbf{Z}_+^n$ and $z \in \mathbf{B}$, we also define

$$K_{z;\alpha}(\zeta) = \frac{\zeta^\alpha}{(1 - \langle \zeta, z \rangle)^{n+1+|\alpha|}},$$

$\zeta \in \mathbf{B}$. Note that $K_z = K_{z;0}$ for every $z \in \mathbf{B}$.

Proposition 3.6. *Let Γ be a separated set in \mathbf{B} and suppose that $\{c_u : u \in \Gamma\}$ is a bounded set of complex coefficients. Then for every pair of $\alpha \in \mathbf{Z}_+^n$ and $z \in \mathbf{B}$, we have*

$$\sum_{u \in \Gamma} c_u (U_u K_z) \otimes (U_u K_{z;\alpha}) \in \mathcal{T}^{(1)}.$$

Proposition 3.7. *Let Γ be a separated set in \mathbf{B} and let $\{c_u : u \in \Gamma\}$ be a bounded set of complex coefficients. Then for every $w \in \mathbf{B}$ we have*

$$\sum_{u \in \Gamma} c_u k_u \otimes k_{\varphi_u(w)} \in \mathcal{T}^{(1)}.$$

Using Proposition 3.7, we can prove the inclusion $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$ by a routine argument: the compactness of the set $\{w \in \mathbf{B} : \beta(0, w) \leq C\}$, finite covering etc, and, of course, an other application of Lemma 2.6.

Analogue of Theorem 1.5 on the Fock space

Improving the notion of localization introduced by X. and Zheng in 2013, Isralowitz, Mitkovski and Wick also introduced weakly localized operators on the Fock space in their 2014 paper:

Definition. A bounded operator T on $H^2(\mathbf{C}^n, d\mu)$ is said to be **weakly localized** if it satisfies the conditions

$$\sup_{z \in \mathbf{C}^n} \int |\langle Tk_z, k_w \rangle| dV(w) < \infty,$$

$$\sup_{z \in \mathbf{C}^n} \int |\langle T^* k_z, k_w \rangle| dV(w) < \infty,$$

$$\lim_{r \rightarrow \infty} \sup_{z \in \mathbf{C}^n} \int_{|z-w| \geq r} |\langle Tk_z, k_w \rangle| dV(w) = 0,$$

$$\lim_{r \rightarrow \infty} \sup_{z \in \mathbf{C}^n} \int_{|z-w| \geq r} |\langle T^* k_z, k_w \rangle| dV(w) = 0.$$

It is easy to see that any sufficiently localized operator is weakly localized.

Replacing the class \mathcal{A}_s by the class of operators in the above Definition, one can prove the analogue of Theorem 1.5 on the Fock space $H^2(\mathbf{C}^n, d\mu)$. The proof is in fact easier in the Fock space case. This is because, compared with the Bergman space, the structure of the Fock space is much simpler, and one generally gets much better “decaying rate” in estimates.

For example, instead of general separated sets, in the Fock space setting we only need to be concerned with the standard lattice \mathbf{Z}^{2n} and its translations. And the **covering** scheme $\mathbf{B} = \cup_{u \in \mathcal{L}} D(u, 2)$ (with overlap) used in the proof of Theorem 1.5 can be replaced by the simple **tiling** scheme

$$\bigcup_{u \in \mathbf{Z}^{2n}} \{u - S\} = \mathbf{C}^n$$

(without overlap), where S is the **fundamental unit cube** in \mathbf{C}^n .

More to the point, the Toeplitz operator T_Φ used in the proof of Theorem 1.5 can be simply replaced by

$$1 = \int k_z^{\text{Fock}} \otimes k_z^{\text{Fock}} dV(z).$$

Thus one does not need the C^* -algebraic argument for the Fock space case. (Thinking about L^p , anyone?)

There is, however, one technical issue in the Fock space case that warrants mentioning. This stems from the fact that there are no bounded analytic functions on \mathbf{C}^n other than constants. Thus the straightforward analogue of Lemma 2.6 on $H^2(\mathbf{C}^n, d\mu)$, while true, is not very useful. In the Fock-space setting, the supremum norm $\|\cdot\|_\infty$ in Lemma 2.6 must be replaced by something else.

Definition 4.3. For an analytic function h on \mathbf{C}^n , we write

$$\|h\|_* = \left(\int |h(\zeta)|^2 e^{-(1/2)|\zeta|^2} dV(\zeta) \right)^{1/2}.$$

Let \mathcal{H}_* be the collection of analytic functions h on \mathbf{C}^n satisfying the condition $\|h\|_* < \infty$.

For each $z \in \mathbf{C}^n$, let U_z be the unitary operator defined by the formula

$$(U_z f)(\zeta) = f(z - \zeta) k_z(\zeta), \quad \zeta \in \mathbf{C}^n,$$

$$f \in H^2(\mathbf{C}^n, d\mu).$$

The following is what replaces Lemma 2.6 in the Fock space case:

Lemma 4.4. *There is a constant $0 < C_{4.4} < \infty$ such that the following estimate holds: Let $\{e_u : u \in \mathbf{Z}^{2n}\}$ be any orthonormal set and let $h_u \in \mathcal{H}_*$, $u \in \mathbf{Z}^{2n}$, be functions satisfying the condition $\sup_{u \in \mathbf{Z}^{2n}} \|h_u\|_* < \infty$. Then*

$$\left\| \sum_{u \in \mathbf{Z}^{2n}} (U_u h_u) \otimes e_u \right\| \leq C_{4.4} \sup_{u \in \mathbf{Z}^{2n}} \|h_u\|_*.$$