The Dirichlet Space as a Quotient

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- For each RKHS, define a distance function δ.
- Discuss the local geometry associated to δ.
- Use that geometry to outline the answer to the question.
- Mention some other places δ shows up.

A RKHS on the Disk is

- A RKHS, H, is a Hilbert space of functions on X for which the point evaluations are continuous. Thus there are kernel functions:

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  - More generally; D_α, $\alpha > 0$, $k_z(w) = (1 - \bar{z}w)^{-\alpha}$, a scale which includes the Hardy and Bergman spaces. The Dirichlet space is the limiting case $\alpha \to 0$.

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- Various Besov-Sobolev spaces; those norms often involve integrals of function and its derivatives. $\mathcal{D}$ and the $DA(k)$ are examples.
- Various spaces of functions which are not holomorphic. For instance the harmonic Bergman space.

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- In which case is $\mathcal{D}$ a quotient of $DA(k)$.

$$\begin{aligned} \mathcal{D} &= DA(k)|_{\Phi(\mathbb{B}^1)} = DA(k)/V(\Phi(\mathbb{B}^1)) \\ &= DA(k) \ominus V(\Phi(\mathbb{B}^1)) = V(\Phi(\mathbb{B}^1))^{\perp}? \end{aligned}$$

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- They are an important subclass of RKHS. They have additional structure which mirrors some of the fine structure of the Hardy space.
- The class includes the Hardy and Dirichlet space, but not the Bergman space. The class includes the D_α, $0 \leq \alpha \leq 1$ and all the $DA(k)$. It does not include the Bergman or Hardy spaces in several variables, nor the D_α, $\alpha > 1$.

The Structure Theorem

- Rather than systematically develop the theory RKHS with the CNPP, I will quote a fundamental descriptive theorem.

Theorem

(Modulo some details) H has the CNPP if and only if (embed) holds; that is, if and only if $\exists\Phi : X \to \mathbb{B}^k$ *so that....*

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- In this theorem **the value $k = \infty$ is allowed.**

A Map is Easy to Find for the Dirichlet Space.

- The monomials are an orthogonal basis; hence $k_{\mathcal{D}}(x,y)$ is a power series in $x\bar{y}$. Set

$$1-\frac{1}{k_{\mathcal{D}}(x,y)}=\sum_{1}^{\infty}\beta_{n}\left(x\bar{y}\right)^{n} \qquad \text{(PD)}$$

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- A crucial fact is that the (PD) is **positive definite**. For $\mathcal{D}$ this follows from the (not totally obvious) positivity of the $\beta' s$.
- If k is not diagonalized by the monomials, this simple approach is lost and the possible positivity of the left hand side of (PD) is more mysterious.

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- If $k_z(w) = (1 - \mathrm{Re}(\bar{z}w))^{-1}$ then $k = 2$ works. The map Φ is $\Phi(x + iy) = (x, y)$. (Note that there is no holomorphy in this example.)

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- $\mathcal{D}$ is simple and explicit, yet quite different from $H^2 = DA(1)$, so it seems a natural test case. (I learned the question form Davidson and Shalit)
- Could approach the question algebraically by considering the rank of an associated positive definite form in (PD). For now we focus more on the Hilbert space geometry.

A Distance Function

- Given a RKHS H of functions on X, define, for $x, y \in X$,

$$\delta_H(x,y) = \sqrt{1 - \left| \left\langle \frac{k_x}{\|k_x\|}, \frac{k_y}{\|k_y\|} \right\rangle \right|^2} = |\sin \measuredangle(k_x, k_y)|$$

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- Or can see it is a metric by showing that

$$\delta(x,y) = \|P_x - P_y\|_{\texttt{operator}},$$

where P_x is projection on $\mathbb{C}k_x$. The first step is to note that $P_x - P_y$ is a rank two self adjoint operator of trace zero. The computation is then mechanical.

Properties of the Distance

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  - Duren and Wier, TAMS 2007

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- I am trying to understand how to extract and apply that information.
- I will discuss how δ is used on the question about embedding. I will also mention some other things about δ.

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- The existence of such m for a generalized version of (SP) is the defining characteristic of spaces with the CNPP.

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- The infinitesimal version of the pseudohyperbolic metric, δ_{H^2}, is the hyperbolic (Poincare-Bergman), the conformal Riemannian metric $\rho_{H^2}(z)\,|dz|$ with $\rho_{H^2}(z) = 2(1-|z|^2)^{-1}$. The associated geodesic distance is the classical Poincare distance

$$d(x,y) = \log \frac{1+\rho_{H^2}(x,y)}{1-\rho_{H^2}(x,y)}$$

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- In particular $\rho_{D_\alpha} = \sqrt{\alpha}\rho_{H^2}$

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- $\mathcal{D}$ is the limiting case of D_α as α tends to zero. What happens to κ?
- We can start with the kernel for $\mathcal{D}$ and then compute using (ρ) and then (κ). The result is a mess but the computer suggests that near the boundary
$$\kappa_{\mathcal{D}}(z) = \log(1 - |z|^2) + O(1)$$
which tends to $-\infty$.

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- For $z \in \mathbb{D}$ let $C(r,z)$ be the circumference of the geodesic circle centered at z having radius r. In flat space $C(r,z){=}2\pi r$. In curved space that is true asymptotically for small r and the next term is determined by the curvature. For fixed z and $r \to 0$

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- In particular, if $\kappa(z) << 0$....

Theorem and Proof Discussion

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  - That is, with respect to the δ metrics Φ must be an isometry.
  - Because Φ is an isometry, it maps the circle $C_r = \{z \in \mathbb{B}^1 : |z| = r\}$ isometrically (in the appropriate metrics) into $\Gamma_\rho = \{z \in \mathbb{B}^1 : |z| = \rho\}$.

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- Suppose x and $y = xe^{i\Delta\theta}$ are on C_r. If the change in angle, $\Delta\theta$, is very small then
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$$A = \delta_{\mathcal{D}}(x, y) \sim \frac{2^n}{n}\Delta\theta$$
- Hence, if we place points on C_r with $\delta_{\mathcal{D}}$ spacing approximately A, then the number of them is roughly
$$O\left(\frac{2\pi}{\Delta\theta}\right) = O\left(\frac{2\pi}{An/2^n}\right) = O\left(\frac{2^n}{An}\right)$$

More Estimates

- These points will be mapped by Φ to a set of points in $\mathbb{B}^1$ which lie on Γ_ρ; and the radii of the two circles must "match"

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- Hence the maximal number of points on D_1 with δ_{H^2} spacing at least A is

$$O\left(\frac{2\pi}{\Delta\theta}\right) = O\left(\frac{n}{A}\right)$$

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- Thus we need to place $O\left(2^n/nA\right)$ points at distance A from each other. But we only have room for $O\left(n/A\right)$.

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- A is fixed and n is at our disposal, so this is impossible.
- This contradiction shows there is no such map for $k = 1$.
- For general k there are at most $O\left(n^{p(k)}/A\right)$ points of spacing at least A on the pseudohyperbolic ball in $\mathbb{B}^k$. We then get a similar contradiction, this time from the fact that $2^n/n$ grows more rapidly than the fixed power $n^{p(k)}$.

For Other "Dirichlet Type" Spaces

- One can try the same proof for D_α, $0 < \alpha < 1$. It shows that you can't embed in $\mathbb{B}^k$ unless $k > k(\alpha)$.

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- But the proof only used the fact that the Φ image of C_r sits on the surface of a sphere Γ_ρ. The image is actually much more constrained than that, but I don't see yet how to use that information.

Other Places the Distance Function Shows Up

- Using the extremal functions for the generalized Schwarz Pick lemma one can mimic the construction of Blaschke products and prove, for instance, that if $\{z_i\} \subset \mathbb{B}^1$ satisfies $\prod \delta_{\mathcal{D}}(0, z_i) < \infty$ the $\{z_i\}$ is a zero set for $\mathcal{D}$; a classic result of Shapiro and Shields. (Note that $\prod \delta_{H^2}(0, z_i) < \infty$ is the Blaschke condition.)

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- If H has a CNPK then δ_H is the (restriction to X) of the Gleason metric on the spectrum of the multiplier algebra.
- The distance function δ_H arises naturally in the descriptions of and perturbation theory for interpolating sequences and sampling sequences. See the book of Seip or the papers of ARSW.

Thank You !