Rational dilation and the Neil parabola

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Some definitions

- $\mathbb{D}$ denotes the unit disk in the complex plane and $\overline{\mathbb{D}}$ its closure.
- The disk algebra, $\mathbb{A}(\mathbb{D})$, is the closure of analytic polynomials in $C(\overline{\mathbb{D}})$, the space of continuous functions on $\overline{\mathbb{D}}$ with the supremum norm.
- The *Neil algebra* is the subalgebra of the disk algebra given by

$$\mathcal{A} = \{f \in \mathbb{A}(\mathbb{D}) : f'(0) = 0\} = \mathbb{C} + z^2\mathbb{A}(\mathbb{D}).$$

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- Constrained algebras, of which $\mathcal{A}$ is one of the simplest examples, are of current interest as a venue for function theoretic operator theory.
- The *Neil parabola* is the distinguished variety given by $W = \{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$ in $\mathbb{C}^2$. It is a manifold except near the origin, where it has a cusp.
- Write $R(W)$ for the algebra of rational functions in two variables with poles off of W.
- The mapping from $R(W)$ to the Neil algebra $\mathcal{A}$ sending $p(z, w)$ to $p(t^2, t^3)$ is a (complete) isometry.

The rational dilation problem

- The *Sz.-Nagy dilation theorem* states that on a Hilbert space, every contraction operator (ie, operator norm less than or equal to 1) dilates to a unitary operator.
- Unitary operators are normal operators with spectrum contained in the boundary of $\mathbb{D}$; that is, $\mathbb{T}$.
- A corollary of the Sz.-Nagy dilation theorem is the *von Neumann inequality*, which implies that T is a contraction if and only if $\|p(T)\| \leq \|p\|$ for every polynomial p, where $\|p\|$ is the again the norm of p in $C(\overline{\mathbb{D}})$.

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- Given a compact subset X of $\mathbb{C}^d$, let $R(X)$ denote the algebra of rational functions with poles off of X with the norm $\|r\|_X$ equal to the supremum of the values of $|r(x)|$ for $x \in X$.
- The set X is a *spectral set* for the commuting d-tuple T of operators on the Hilbert space H if the spectrum of T lies in X and $\|r(T)\| \leq \|r\|_X$ for each $r \in R(X)$ (that is, a version of the von Neumann inequality holds).
- If N is also a d-tuple of commuting operators with spectrum in X and acting on the Hilbert space K, then T *dilates* to N provided there is an isometry $V : H \to K$ such that $r(T) = V^* r(N) V$ for all $r \in R(X)$.

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- *Rational dilation problem*: If X is a spectral set for T does T dilate to a tuple N of commuting normal operators with spectrum in the Shilov boundary of X relative to the algebra $R(X)$?

Known positive cases

- The Sz.-Nagy dilation theorem is just the statement that rational dilation holds for the closed disk.
- Foias and Lebow extended this to more general simply connected planar domains.
- Jim Agler showed that rational dilation holds for annuli.
- Andô's theorem is a two variable version of the Sz.-Nagy dilation theorem. Hence rational dilation holds on the bidisk $\mathbb{D}^2$ in $\mathbb{C}^2$.
- Jim Agler and Nicholas Young showed that rational dilation holds for the symmetrized bidisk.

Known negative cases

- An example due to Parrott shows that rational dilation fails for $\mathbb{D}^d$ for $d > 2$.
- Using computer algebra methods, Agler, Harland and Rafael showed that rational dilation fails for the unit disk with two particular smaller disks removed (a triply connected domain).
- Dritschel and McCullough showed that it fails for all compact triply connected regions with smooth boundary components.
- Pickering extended this to compact planar regions with higher connectivity as long as the Schottky double is hyperelliptic (automatic in the triply connected case).
- Sourav Pal has shown that it also fails for the tetrablock.

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- A tuple T acting on the Hilbert space H with spectrum in X determines a unital representation of π_T of $R(X)$ on H via $\pi_T(r) = r(T)$.
- The condition that X is a spectral set for T is equivalent to the condition that this representation is *contractive*.
- A representation π of $R(X)$ is *completely contractive* if for all n and all $F \in M_n(R(X))$, $\pi^{(n)}(F) := (\pi(F_{i,j}))$ is contractive, the norm of F being given by $\|F\|_\infty = \sup\{\|F(x)\| : x \in X\}$ with $\|F(x)\|$ the operator norm of $F(x)$.

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- Arveson proved that T dilates to a tuple N of normal operators with spectrum in the (Shilov) boundary of X (with respect to $R(X)$) if and only if π_T is completely contractive.
- *Reformulated rational dilation problem:* Is every contractive representation of $R(X)$ completely contractive?

The Neil algebra and the Neil parabola

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- Hence any (completely) contractive representation of $\mathcal{A}$ induces a (completely) contractive representation of $R(W)$, and dilations translate as well.
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- So if we can solve the reformulated rational dilation problem for $\mathcal{A}$, this will lead to a solution of the rational dilation problem for $R(W)$.
- Studying the rational dilation problem on such varieties helps us to begin to understand more generally why rational dilation holds for some sets and not for others.

Completely contractive representations of $\mathcal{A}$

- As a (unital) Banach algebra, $\mathcal{A}$ is generated by the functions z^2 and z^3.
- Hence any bounded unital representation is determined by its values on these two functions. If $\pi : \mathcal{A} \to B(H)$ is a bounded representation, $X = \pi(z^2)$ and $Y = \pi(z^3)$, then X, Y are commuting operators which satisfy $X^3 = Y^2$.
- If we further insist that π is contractive, then X and Y are contractions

Theorem 1 (Broshinski).

A representation $\pi : \mathcal{A} \to B(H)$ is completely contractive if and only if there is a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$ such that for all $n \geq 0$, $n \neq 1$,

$$\pi(z^n) = P_H U^n|_H. \tag{1}$$

Example

- Let K be a separable Hilbert space with orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$, and let U be the bilateral shift.
- Let $H \subset K$ be defined as $H = e_0 \vee \bigvee_{n=2}^{\infty} e_n$.
- H is invariant for U^2 and U^3, and so by the above theorem, π given by $\pi(z^n) = P_H U^n|_H = U^n|_H$, $n \geq 0$, $n \neq 1$, is a completely contractive representation of $\mathcal{A}$.
- Suppose there were some $T \in B(H)$ with $T^2 = \pi(z^2)$ and $T^3 = \pi(z^3)$.
- Then $e_3 = U^3 e_0 = \pi(z^3)e_0 = \pi(z^2)Te_0$.
- However, $\langle \pi(z^2)e_n, e_3 \rangle = \langle U^2 e_n, e_3 \rangle = 0$ for $n \geq 0$, $n \neq 1$, and hence e_3 is orthogonal to the range of $\pi(z^2)$.
- Hence there is no way to define Te_0 so that $e_3 = \pi(z^2)Te_0$, and so there can be no such T.

A set of test functions for $\mathcal{A}$

- For $\lambda \in \mathbb{D}$, let
$$\varphi_\lambda(z) = \frac{z - \lambda}{1 - \lambda^* z},$$
and
$$\psi_\lambda(z) = z^2 \varphi_\lambda(z).$$
Write ∞ for the point at infinity in the one point compactification $\mathbb{D}_\infty$ of $\mathbb{D}$ and set $\psi_\infty = z^2$.
- The set
$$\Psi = \{\psi_\lambda : \lambda \in \mathbb{D}_\infty\},$$
with the topology and Borel structure inherited from $\mathbb{D}_\infty$ is called a set of *test functions*.
- That is, for any $x \in \mathbb{D}$, the $\sup_{\psi \in \Psi} |\psi(x)| < 1$ and the elements of Ψ separate the points of $\mathbb{D}$.

Agler decomposition and minimality of the set of test functions

Theorem 2 (Dritschel, Pickering).

An analytic function f in the disk belongs to $\mathcal{A}$ and satisfies $\|f\|_\infty \leq 1$ if and only if there is a positive kernel $\mu \in M^+(\mathbb{D}) = \{\mu : \mathbb{D} \times \mathbb{D} \to M(\Psi)\}$, $M(\Psi)$ be the space of finite Borel measures on Ψ, such that

$$1 - f(x)f(y)^* = \int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{2}$$

for all $x, y \in \mathbb{D}$.

Furthermore, Ψ is minimal, in the sense that there is no proper closed subset of $E \subset \Psi$ such that for each $f \in \mathcal{A}$, there exists a positive kernel μ of finite Borel measures supported on E such that

$$1 - f(x)f(y)^* = \int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{3}$$

A Kaiser-Varopoulos type example

- Recall that Kaiser and Varopoulos first showed that there exist three commuting contractions (T_1, T_2, T_3) such that the unital representation π of $R(\mathbb{D}^3)$ given by $\pi(z_j) = T_j$ is not contractive.

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- Coming back to the last theorem, for $E \subset \Psi$ a closed subset, let $C_{1,E}$ denote the cone consisting of the kernels

$$\left(\int_E (1 - \psi(x)\psi(y)^*) \, d\mu_{x,y}(\psi) \right)_{x,y \in \mathbb{D}} . \tag{4}$$

- In particular, if we choose $E = \{z^2, z^3\}$, it follows from this theorem that there exists a function $f \in \mathcal{A}$ with $\|f\|_\infty \leq 1$ such that $1 - f(x)f(y)^* \notin C_{1,E}$.

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We therefore have

Corollary 3.

There exists a pair of commuting contractive matrices X, Y with $X^3 = Y^2$, but such that the representation of $\mathcal{A}$ determined by $\pi(z^2) = X$, $\pi(z^3) = Y$ is not contractive.

How to disprove rational dilation

- Given $F \in M_2(\mathcal{A})$, let $\Sigma_{F,\mathcal{F}}$ denote the kernel

$$\Sigma_{F,\mathcal{F}} = (1 - F(x)F(y)^*)_{x,y\in\mathcal{F}}.$$

- Let I denote the ideal of functions in $\mathcal{A}$ which vanish on $\mathcal{F}$. Write $q : \mathcal{A} \to \mathcal{A}/I$ for the canonical projection, which is completely contractive.

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Lemma 4.

If $F \in M_2(\mathcal{A})$ with $\|F\| \leq 1$, but $\Sigma_{F,\mathcal{F}} \notin C_{2,\mathcal{F}}$, then there exists a Hilbert space H and representation $\tau : \mathcal{A}/I \to B(H)$ such that $\tau \circ q$ is a contractive, but not 2-contractive $($and hence not completely contractive$)$ representation of $\mathcal{A}$.

The counterexample for the Neil algebra

- Arveson's theorem tells us that for rational dilation to hold, contractive representations must be completely contractive.
- Roughly speaking, this will imply that when rational dilation holds, any matrix valued function on which the von Neumann inequality holds must "diagonalise", thus reducing the matrix case back to the scalar case.

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- For the Neil algebra, set

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & \varphi_2 \end{pmatrix},$$

  where $\frac{1}{\sqrt{2}} U$ is a 2×2 unitary matrix with all non-zero entries. To be concrete, choose

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

- Then Φ is a 2×2 matrix inner function with $\det \Phi(\lambda) = 0$ at precisely the two nonzero points λ_1 and λ_2.
- The function

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  is in $M_2(\mathcal{A})$ and is a rational inner function, so $\|F\|_\infty = 1$.
- Ultimately we identify a set of six points $\mathcal{F}$ which is a set of uniqueness for F, and show that for this set, $\Sigma_{F,\mathcal{F}} \notin C_{2,\mathcal{F}}$ since it cannot be diagonalised by our choice of U.