The Nullstellensatz and the Corona Theorem

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When does the polynomial equation

$$p_1(z), p_2(z), \cdots p_m(z) \in \mathbb{C}[z],$$

$$p_1(z) = p_2(z) = \cdots = p_m(z) = 0$$

have no solutions in $\mathbb{C}^n$, where $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$?

This question is equivalent to $V(p_1, p_2, \cdots, p_m) = \emptyset$, where

$$V(p_1, p_2, \cdots, p_m) := \{z \in \mathbb{C}^n : p_i(z) = 0, \forall 1 \leq i \leq m\}.$$

If there exist polynomials $q_1(z), q_2(z), \cdots, q_m(z) \in \mathbb{C}[z]$ such that

$$p_1(z)q_1(z) + p_2(z)q_2(z) + \cdots + p_m(z)q_m(z) = 1,$$

for all $z \in \mathbb{C}^n$, then $V(p_1, p_2, \cdots, p_m) = \emptyset$.

example:

$$p_1(z) = 1 + z^2$$
$$p_2(z) = 1 + z^2 + z^4$$

Obviously, $p_1(i) = p_1(-i) = 0$, but $p_2(i) = p_2(-i) = 1$ so that p_1 and p_2 have no common zero. This means that $V(p_1, p_2) = \emptyset$.

Note that

$$-z^2(1 + z^2) + (1 + z^2 + z^4) = 1.$$

Theorem (Hilbert's Weak Nullstellensatz)

$$V(p_1, p_2, \cdots, p_m) = \emptyset$$

is equivalent to the existence of

$$q_1, q_2, \cdots, q_m \in \mathbb{C}[z]$$

with

$$p_1 q_1 + p_2 q_2 + \cdots p_m q_m = 1.$$

Question: What is the maximum degree of the q_i?

Let d_i denote the degree of p_i. Without loss of generality, we can assume that $d_1 \geq d_2 \geq \cdots \geq d_m$.

1. (G. Hermann, 1926) $2(2d_1)^{2^{n-1}}$
2. (W. B. Brownawell, 1987) $\min\{n, m\} n d_1^{\min\{n,m\}} + \min\{n, m\} d_1$
3. (J. Kollar, 1988) $\deg p_i q_i \leq \max\{3, d_1\}^{\min\{n,m\}}$
4. (M. Sombra, 1999) $\deg p_i q_i \leq 2 d_m \prod_{j=1}^{\min\{n,m\}-1} d_j$

In this talk, only univariate polynomials will be considered, i.e., when $n = 1$.

Theorem (H. Kwon, A. Netyanun, and T. T. Trent)

Let $p_j \in \mathbb{C}[z]$ *be* m *polynomials with no common zero in* $\mathbb{C}$ *and let* $\deg p_j = d_j$, *where* $d_1 \geq d_2 \geq \cdots \geq d_m$. *Then there exist* m *polynomials* $q_j \in \mathbb{C}[z]$ *such that*

$$\sum_j p_j q_j = 1$$

with the bound

$$\deg q_j \leq d_1 - 1.$$

Let $p_1, p_2 \in \mathbb{C}[z]$ with no common zero in $\mathbb{C}$. Let F be the 1×2 matrix, $F = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$. Then

$$F \frac{F^*}{FF^*} = 1,$$

but we want $\frac{F^*}{FF^*}$ to be a matrix of polynomials. Following the idea of L. Hörmander, we find Q and G with the property that

$$\bar{\partial}\Big(\frac{F^*}{FF^*} - QG\Big) = 0,$$

where $\operatorname{ran} Q(z) = \ker F(z)$ for all $z \in \mathbb{C}$ so that

$$F(\frac{F^*}{FF^*} - QG) = 1$$

still holds. Here, $\bar{\partial}$ stands for the differential operator

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

$\underline{FQ = 0}$

For $F = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$, let

$$Q = \begin{bmatrix} -p_2 \\ p_1 \end{bmatrix}.$$

For $F = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$, let

$$Q = \begin{bmatrix} -p_2 & -p_3 & 0 \\ p_1 & 0 & -p_3 \\ 0 & p_1 & p_2 \end{bmatrix}.$$

For $F = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix}$, let

$$Q = \begin{bmatrix} -p_2 & -p_3 & -p_4 & 0 & 0 & 0 \\ p_1 & 0 & 0 & -p_3 & -p_4 & 0 \\ 0 & p_1 & 0 & p_2 & 0 & -p_4 \\ 0 & 0 & p_1 & 0 & p_2 & p_3 \end{bmatrix}.$$

Note that since

$$\bar{\partial}\Big(\frac{F^*}{FF^*}\Big) = \frac{(F')^*}{FF^*} - \frac{F^*F(F')^*}{(FF^*)^2},$$

G must satisfy the equation

$$\bar{\partial}G = \frac{Q^*(F')^*}{(FF^*)^2}.$$

We take G to be the Cauchy transform of $\frac{Q^*(F')^*}{(FF^*)^2}$.

$$(\mathcal{C}f)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{w-z}dA(w),$$

In order to show that $\frac{F^*}{FF^*} - QG$ is a matrix of polynomials, we use

Lemma

Let $F(z)$ be a $1 \times n$ matrix of polynomial entries $p_j(z) \in \mathbb{C}[z]$. Then there exists an $R \geq 1$ dependent on the p_j such that for $|z| \geq R$,

$$F(z)F^*(z) \geq C|z|^{2d_1},$$

where d_1 denotes the maximum degree of the p_j.

Lemma

Let $Y(z)$ be an entire function on $\mathbb{C}$ such that

$$\lim_{|z|\to\infty} \frac{|Y(z)|}{|z|^N} = 0$$

for some fixed positive integer N. Then $Y(z)$ is a polynomial of degree at most $N-1$.

Theorem (H. Kwon, A. Netyanun, and T. T. Trent)

Let $F(z) = [p_{ij}(z)]$ be an $m \times n$ matrix of polynomials $p_{ij}(z) \in \mathbb{C}[z]$ satisfying the corona condition

$$F(z)F(z)^* \geq \epsilon > 0,$$

for all $z \in \mathbb{C}$. We can assume without loss of generality that the k_i, the maximum degree of the entries in the i-th row of $F(z)$, are in increasing order. Then there exists $G(z) = [q_{ij}(z)], q_{ij}(z) \in \mathbb{C}[z]$, an $n \times m$ matrix of polynomials such that $F(z)G(z) = I_m$. Moreover, if $1 \leq k_1$, then the degree bound for the $q_{ij}(z)$ is given by

$$(2m-1)k_m + 2k_{m-1} + 4k_{m-2} + \cdots + 2(m-1)k_1 - m.$$

Otherwise, if l is the smallest positive integer such that $1 \leq k_{l+1}$, then the degree bound for the $q_{ij}(z)$ is given by

$$2k_m + 4k_{m-1} + \cdots + 2(m-l)k_{l+1} - (m-l).$$

Denote by $p_i(z), 1 \le i \le m$, the i-th row of $F(z)$. We want to find m column vectors $q_i(z)$ such that

$$F(z)q_i(z) = e_i.$$

To find $q_1(z)$, we need to solve

$$p_i(z)q_1(z) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } 2 \le i \le m. \end{cases}$$

As a result of the previous theorem, we can find $v_1(z)$ such that

$$p_1(z)v_1(z) = 1.$$

Again, $v_1(z)$ must be modified.

1. $v_2(z) := v_1(z) - Q_{p_1}^{(1)}(z)w_1(z)p_2(z)v_1(z)$ replaces $v_1(z)$, where
$$p_2(z)Q_{p_1}^{(1)}(z)w_1(z) = 1.$$
2. $v_3(z) := v_2(z) - Q_{p_1}^{(1)}(z)Q_{p_2}^{(2)}(z)w_2(z)p_3(z)v_2(z)$, replaces $v_2(z)$, where
$$p_3(z)Q_{p_1}^{(1)}(z)Q_{p_2}^{(2)}(z)w_2(z) = 1.$$

The degree bound for $w_1(z)$ equals $k_1 + k_2 - 1$ and hence the degree bound for $v_2(z)$ equals
$$k_1 + (k_1 + k_2 - 1) + k_2 + (k_1 - 1).$$

For $F = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$, let

$$Q_F^{(1)} = \begin{bmatrix} -p_2 \\ p_1 \end{bmatrix}.$$

For $F = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$, let

$$Q_F^{(1)} = \begin{bmatrix} -p_2 & -p_3 & 0 \\ p_1 & 0 & -p_3 \\ 0 & p_1 & p_2 \end{bmatrix},$$

and

$$Q_F^{(2)} = \begin{bmatrix} p_3 \\ -p_2 \\ p_1 \end{bmatrix}.$$

For $F = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix}$, let

$$Q_F^{(1)} = \begin{bmatrix} -p_2 & -p_3 & -p_4 & 0 & 0 & 0 \\ p_1 & 0 & 0 & -p_3 & -p_4 & 0 \\ 0 & p_1 & 0 & p_2 & 0 & -p_4 \\ 0 & 0 & p_1 & 0 & p_2 & p_3 \end{bmatrix},$$

$$Q_F^{(2)} = \begin{bmatrix} p_3 & p_4 & 0 & 0 \\ -p_2 & 0 & p_4 & 0 \\ 0 & -p_2 & -p_3 & 0 \\ p_1 & 0 & 0 & p_4 \\ 0 & p_1 & 0 & -p_3 \\ 0 & 0 & p_1 & p_2 \end{bmatrix},$$

and

$$Q_F^{(3)} = \begin{bmatrix} -p_4 \\ p_3 \\ -p_2 \\ p_1 \end{bmatrix}$$

THANK YOU VERY MUCH!!