Dixmier trace of quotient module on bounded symmetric domain

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Outline

Dixmier trace

For $T \in K(H)$, singular value sequence (counted with multiplicity):

$$s_j(T) = \lambda_j(|T|), s_1 \geq s_2 \geq s_3 \geq ...$$

Schatten p-class $\mathcal{L}^p(H) : \sum_j s_j^p(T) < +\infty$

Macaev class $\mathcal{L}^{p,q}(H)$: the interpolation space of $\mathcal{L}^1(H)$ and $K(H)$
For the extreme case $q = \infty$, $T \in \mathcal{L}^{p,\infty}(H)$ if

$$\sum_{i=1}^{n} s_i(T) = O(\log n), \text{ when } p = 1; \quad s_n(T) = O(n^{-1/p}), \text{ when } p > 1.$$

Dixmier trace

For compact positive operator T,

$T \in \mathcal{L}^1(H)$, trace

$$trT = \sum_j s_j(T)$$

$T \in \mathcal{L}^{1,\infty}(H)$ measurable, Dixmier trace

$$tr_\omega(T) = \lim_{n\to\infty} \frac{\sum_{i=1}^n s_i(T)}{\log n}$$

independent of ultrafilter ω on l^∞.

Dixmier trace

$\Omega \subseteq \mathbb{C}^n$ bounded strongly pseudoconvex domain,

$$P : L^2(\Omega) \to H^2(\Omega) \ \textit{orthogonal pojection}$$

$$T_f\phi = P(f\phi), f \ \textit{bounded}$$

$n = 1, \Omega = \mathbb{D}$ unit disk, f, g analytic on $\overline{\Omega}$

$$tr[T_f^*, T_g] = \int_{\mathbb{D}} dz\overline{f'(z)}g'(z).$$

For $n > 1$, $\Omega = \mathbb{B}_n$, and $f_1, g_1, \cdots, f_n, g_n$ analytic on $\overline{\Omega}$

$$[T_{f_1}^*, T_{g_1}] \cdots [T_{f_n}^*, T_{g_n}] \notin \mathcal{L}^1$$

Dixmier trace

The complete antisymmetric sum $[A_1, ..., A_m]$ is

$$\sum_{\tau \in S_m} \epsilon(\tau) A_{\tau(1)} A_{\tau(1)} \cdots A_{\tau(m)}$$

where S_m is the symmetric group on $[1, 2.....m]$ and ϵ is the signum character on S_m.

Helton-Howe

On unit ball $\mathbb{B}_n$, complete anti-symmetric sum $[T_{f_1}, \cdots, T_{f_{2n}}]$ is in trace class, and

$$tr[T_{f_1}, \cdots, T_{f_{2n}}] = \frac{1}{Vol(\mathbb{B}_n)} \int_{\mathbb{B}_n} df_1 \wedge \cdots \wedge df_{2n}$$

Englis, Rochberg

For Hankel operator with smooth analytic symbol in the Bergman space on the unit disk $\mathbb{D}$,

$$tr_\omega(|H_{\overline{f}}|) = \int_{\mathbb{T}} |f'(e^{i\theta})| \frac{d\theta}{2\pi}$$

Englis-Zhang-Guo ; Englis-Zhang

For $f_1, \cdots, f_{2n} \in C^\infty(\overline{\Omega})$, $[T^*_{f_1}, T_{g_1}] \cdots [T^*_{f_n}, T_{g_n}] \in \mathcal{L}^{1,\infty}$ and

$$tr_\omega[T^*_{f_1}, T_{g_1}] \cdots [T^*_{f_n}, T_{g_n}] = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \prod_j \mathcal{L}(\overline{\partial_b} f_j, \partial_b g_j) \eta \wedge (d\eta)^{n-1}.$$

Dixmier trace

Conjecture of Arveson, Douglas

Homogenous quotient module $[I]^{\perp}$ is p-essentially normal for $p > n = \dim Z(I)$.

Guo, Wang and Zhang:
trace and Dixmier trace formula for operators with analytic polynomial symbol on essentially normal quotient module in $H^2(\mathbb{D}^2)$.

bounded symmetric domain

definition

A bounded domain D in the complex space Z is symmetric: for every $x \in D$, there is an biholomorphism s such that $s^2 = id_D$, and x is an isolated fixed point of s.

Let

$$G := Aut(D) = \{\Phi : D \to D \; biholomorphic\}$$

$$K := \{g \in G : g(0) = 0\},$$

Then $D = G/K$.

Classification:
Four Classical domains and Two exceptional domains

Example

Type I tube domain:
$D = \{z \in M_r(\mathbb{C}) : \|z\| < 1\} = SU_{r,r}(\mathbb{C})/S(U_r(\mathbb{C}) \otimes U_r(\mathbb{C}))$
with the action $\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) := (az+b)(cz+d)^{-1}$

boundary:
$\partial D = \{z \in M_r(\mathbb{C}) : \|z\| = 1\}$
$S_k = \{z \, partial \, isometry : r(z) = k\}$ for $1 \leq k \leq r$.
Then, we have the boudanry decomposition

$$\partial D = \cup_{1 \leq k \leq r} \cup_{c \in S_k} (c + \Omega_c)$$

$S = S_r$ is the Shilov boundary of D.

Hardy space

Let S be the Shilov boundary of the bounded symmetric domain D in the complex space Z. Since K acts transitively on S, there exists unique $K-$invariant probability measure on S. Denote $L^2(S)$ be the space of L^2-integrable functions. The Hardy space

$$H^2(S) = \{\phi \in L^2(S) : \phi \, holomorphic \ on \, D\}$$

is the closure of the algebra $\mathcal{P}(Z)$ of all polynomial on Z.
The natural action of K induces a multiplicity-free Peter Weyl decomposition

$$H^2(S) = \bigoplus_{\lambda} \mathcal{P}_\lambda(Z)$$

where $\lambda = (\lambda_1, \cdots, \lambda_r)$ runs all $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$.

Hardy space

It is well known that D can be realized as the open unit ball of an irreducible hermitian Jordan triple Z. Let $e_1, \cdots, e_r$ be a frame of minimal tripotents of Z, with joint Peirce decomposition

$$Z = \bigoplus_{0 \leq i \leq j \leq r} Z_{ij}$$

Then $\mathcal{P}_\lambda(Z)$ has the highest weight vector

$$N_\lambda(Z) := N_1(z)^{\lambda_1 - \lambda_2} N_2(z)^{\lambda_2 - \lambda_3} \cdots N_r(z)^{\lambda_r}$$

where $N_1, \cdots, N_r$ are the Jordan theoretic minors.
Put

$$a := dimZ_{ij}, 1 \leq i < j \leq r$$

$$b := dimZ_{0j}, 1 \leq j \leq r$$

Hardy space

Set

$$\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b$$

the numerical invariant $p := 2 + a(r-1) + b$ is called the genus of Z.

Consider the polynomial algebra $\mathcal{P}(Z)$ with the Fock inner product

$$(p|q)_Z := (\partial_p q)(0)$$

for all $p, q \in \mathcal{P}(Z)$. The irreducible $K-$type $\mathcal{P}_\lambda(Z)$ has the inner product

$$(p|q)_S = \frac{1}{(d/r)_\lambda}(p|q)_Z$$

for all $p, q \in \mathcal{P}_\lambda(Z)$. Here the Pochhammer symbol $(\mu)_\lambda$ is defined by

$$(\mu)_\lambda = \prod^{r}(\mu - \frac{a}{2}(j-1)_{\lambda_j}$$

Toeplitz algebra

Let

$$P_S : L^2(S) \to H^2(S)$$

denote the Cauchy-Szego projection. For $f \in L^\infty(S)$, we have the Hardy-Toeplitz operator

$$T_f(\phi) = P_S(f\phi)$$

acting on $\phi \in H^2(S)$. Let

$$\mathfrak{T}(S) = C^*(T_f : f \in C(S))$$

denote the Toeplitz C^*-algebra generated by continues symbol function.

Toeplitz algebra

Let P_k be the projection onto the subspace $\oplus_{\lambda_{k+1}=0}\mathcal{P}_\lambda$, then $P_k \in \mathfrak{T}(S)$. Let $\mathfrak{T}_k$ be the C^*-algebra generated by $T_p P_k T_q^*$ with polynomial symbol p, q.

Upmeier

The C^*-algebras form an ascending chain

$$0 \subseteq K(H) = \mathfrak{T}_0 \subseteq \mathfrak{T}_1 \subseteq \cdots \subseteq \mathfrak{T}_r = \mathfrak{T}(S).$$

And there is a C^*-algebra isomorphism

$$\mathfrak{T}_k/\mathfrak{T}_{k-1} = C(S_k) \otimes K(H_k),$$

where H_k is a Hilbert space with $\dim(H_k) = \infty$ for $k < r$ and $\dim(H_k) = 1$ for $k = r$.

quotient module

Let Z_1 be the complex variety of all elements in Z of rank≤ 1. Then

$$n := dim_{\mathbb{C}} Z_1 = 1 + a(r-1) + b = p - 1.$$

And S_1 is the boundary of $Z_1 \cap D$.
The subspace

$$H_1^{\perp}(S) = \{p \in H^2(S) : p|_{Z_1 \cap D} = 0\} = \bigoplus_{\lambda_2 > 0} \mathcal{P}_{\lambda}(Z)$$

is a submodule in $H^2(S)$, with the quotient module

$$H_1(S) = \bigoplus_{m=0}^{\infty} \mathcal{P}_{m,0,\cdots,0}(Z)$$

The orthogonal projection $P = P_1 : H^2(S) \to H_1^2(S)$.
Define

$$S_f := PT_f P.$$

Define $\Lambda(p_m) = mp_m$ for $p_m \in \mathcal{P}_m$. Then $\frac{1}{\Lambda+1} \in \mathcal{L}^{n,\infty}$.

Proof. In fact, for any index λ we have

$$\begin{aligned}\dim \mathcal{P}_\lambda(Z) &= \prod_{j=1}^{r} \frac{(1+b+\frac{a}{2}(r-j))_{\lambda_j}}{(1+\frac{a}{2}(r-j))_{\lambda_j}} \\ &\cdot \prod_{1\le i<j\le r} \frac{\lambda_i-\lambda_j+\frac{a}{2}(r-j)}{\frac{a}{2}(r-j)} \frac{(\frac{a}{2}(j+1-i))_{\lambda_i-\lambda_j}}{(1+\frac{a}{2}(j-i-1)))_{\lambda_i-\lambda_j}}\end{aligned}$$

for $\lambda = (m, 0, \cdots, 0)$ and m large enough we obtain

$$\dim \mathcal{P}_m(Z) \sim c \cdot m^{b+a(r-1)} = cm^{n-1}$$

for some constant $c > 0$ independent of m.

Proposition

$\|[S_u, S_v^*]\Lambda\| \leq C\|u\|\|v\|$ and $\|[S_u, \Lambda]\|$ is bounded for any linear function $u(z) = (z|u), v(z) = (z|v)$.

Proof. Using K-invariant technique, we can reduce to estimate $\|[S_u, S_v^*]\Lambda N_1^m\| \leq C\|u\|\|v\|\|N_1^m\|$ with $N_1(z) = (z|e_1)$.
With some algebraic reductive, it is enough to consider the case rank 2. And we can show in rank 2 that $(z|v)N_1^m$ has the harmonic projection

$$S_v N_1^m = P((z|v)N_1^m) = (z|v)N_1^m - \frac{1}{m'}N(z)(v^\partial N)^\# N_1^m,$$

where N is the determinant, and $m' := m + \frac{a}{2}$.

rank 2

For rank 2 domain D,

$$H^2(S) = H^2(\mathbb{T}) \otimes L^2(S^{n-1})$$

$$P = 1 \otimes Id$$

$$T_u = id \otimes A_u + T_z \otimes A_{\overline{u}}^*$$

So, for real analytic symbol

$$S_{\overline{v}u} = PT_v^* T_u P = S_v^* S_u + S_{\overline{v}} S_{\overline{u}}^*$$

Let P^β be the projection onto $\bigoplus_{m=0}^{\infty} \mathcal{P}_{m,\beta}$, $P^j = \sum_{|\beta|=j} P^\beta$. Let $\mathcal{A}$ be *-algebra generated by S_p, $\mathcal{C}$ *-algebra generated by T_p with polynomial symbol and P^β. Then $\mathcal{B} := \{T \in \mathcal{C} : T\Lambda \; bounded\}$ is a two-side ideal of $\mathcal{C}$.

Proposition

(1). For analytic polynomials p, q, $PT_p^* T_q P \in \mathcal{A} + \mathcal{B}$.
(2). $P^\beta = \sum_i c_i T_{p_i} P T_{p_i}^* + \mathcal{B}$ for some polynomial p_i with $deg(p_i) \leq |\beta|$.

Theorem 1

For real analytic polynomials f, g, we have $[S_f, S_g] \in \mathcal{L}^{n,\infty}$.

K-invariant operator

For a fixed partition λ, choose an orthonormal basis $p_i \in \mathcal{P}_\lambda(Z)$ with respect to the Fock inner product. Then

$$A^\lambda := \sum_i T_{p_i} P T_{p_i}^*$$

is a K-invariant operator, independent of the choice of orthonormal basis. So,

$$A^\lambda = \sum_{\beta \leq \lambda \leq (m,\beta)} c_\beta^\lambda(m) P_{m,\beta}.$$

outline of proof

We use an inductive argument for $m = min(\deg(p), \deg(q)) = |\beta|$. Suppose (1) and (2) hold for $m = k$. Then for any $|\lambda| \leq k + 1$,

$$A^{\lambda} = \sum_{\beta \leq \lambda} c_{\beta}^{\lambda} P^{\beta} + \mathcal{B}.$$

After a careful estimation for the coefficients c_{β}^{λ}, we can reverse the equation and show

$$P^{\beta} = \sum_{\lambda \leq \beta} \tilde{c}_{\lambda}^{\beta} A^{\lambda} + \mathcal{B}$$

outline of proof

we divide the proof of (1) to two step:
step 1: $\deg(p) = \deg(q) = m = k + 1$

$$PT_p^* T_q P = \sum_{|\beta| \leq k+1} PT_p^* P^\beta T_q P.$$

For $|\beta| \leq k < k + 1$, using the representation of P^β,

$$PT_p^* P^\beta T_q P = \sum_i PT_p^* T_{p_i} PT_{p_i}^* T_p P + \mathcal{B}$$

For $|\beta| = k + 1$,

proposition

Let $p, q \in \mathcal{P}(Z)$ such that $min(\deg(p), \deg(q)) \leq s$. Let $|\lambda| = s$. Then

$$PT_p^*(A^{h,\lambda} - c_\lambda^{h,\lambda} P^\lambda) T_q P \in \mathcal{B}.$$

proposition

Let $p \in \mathcal{P}(Z)$ be a polynomial of degree $\leq k$. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ be a partition such that $\beta' = (\beta_2, \ldots, \beta_r)$ satisfies $|\beta'| \geq k$. Then

$$PT_p^* T_q P = PT_{T_p^* q} P$$

for all $q \in \mathcal{P}_\beta(Z)$.

So,

$$PT_p^* P^\beta T_q P = \mathcal{B} + PT_p^* A^{h,\beta} T_q P$$

$$= \mathcal{B} + \sum_i PT_p^* T_{p_i} PT_{p_i}^* T_q P = \mathcal{B} + \sum_i (PT_{T_p^* p_i} P)(PT_{T_q^* p_i}^* P).$$

Step 2:
Use the fact $[T_u, T_v^*]$ is block "diagonal" operator respecting the Peter-Weyl decomposition for linear function u, v and a long algebraic reductive

Dixmier trace formula

The compact manifold S_1, all tripotent of rank 1 is the boundary of the strongly pseudoconvex variety

$$\Omega = D \cap Z_1 = \{z \in Z : rank(z) \leq 1, \|z\| < 1.\}$$

which has its only singularity at 0. Let

$$H^2(S_1) := \{\phi \in L^2(S_1) : \phi\, holomorphicon\, \Omega\}$$

be the associated 'little' Hardy space. Let $\pi : L^2(S) \to H^2(S)$ denote the Cauchy-Szego projection. For $f \in L^\infty(S_1)$ we have the Toeplitz operator

$$\tau_f \phi = \pi(f\phi).$$

Dixmier trace formula

For $m \geq 0$, put

$$\Lambda_m^2 := \frac{(a/2)_m}{(ra/2)_m}$$

Then the mapping $U : H_1^2(S) \to H^2(S_1)$, defined by

$$Up := \Lambda_m p|_{S_1}$$

for $p \in \mathcal{P}_m$ is a unitary isomorphism. For $f(z) = (z|u)$, with restriction $f|_{S_1}$, we have

$$US_f U^* = \frac{\Lambda_{m+1}}{\Lambda_m} \tau_{f|_{S_1}}$$

and

$$US_f U^* - \tau_{f|_{S_1}} \in \mathcal{L}^{n,\infty}$$

proposition

For a real polynomial $\overline{p}q$ on Z, we have that

$$US_f U^* - \tau_h \in \mathcal{L}^{n,\infty},$$

where $h(e) = (p_e|q_e)_{S_e}$.

Proof. For such f,

$$US_f U^* = \tau_{\sum_i \overline{p_i} q_i} + \mathcal{B}.$$

Using the symbol map, for any $e \in S_1$,

$$(\sigma_1 S_f)(e) = (1_e \otimes 1_e) T^*_{p_e} T_{q_e} (1_e \otimes 1_e) = (p_e|q_e)_{S_e} (1_e \otimes 1_e),$$

and

$$(\sigma_1 [\sum_i S^*_{p_1} S_{q_i}])(e) = \sum_i \overline{p_i}(e) q_i(e) (1_e \otimes 1_e).$$

Then $h(e) = \sum_i \overline{p_i}(e) q_i(e) = (p_e|q_e)_{S_e}$ on S_1.

Dixmier trace formula

There is a harmonic extension operator $\mathcal{P}$ for $C(S)$ to a function on D given by a Poisson integral

$$\mathcal{P}f(z) = \int_S ds \mathcal{P}(z,s) f(s) = \int_S ds \frac{\triangle(z,z)^{d/r}}{\triangle(z,s)^{d/r}\triangle(s,z)^{d/r}} f(s)$$

Koranyi, Schlichtkrull

For $f \in C(S)$, we have that $\mathcal{P}f \in C(\overline{\Omega})$. And for each face $\overline{c+\Omega_c}$, $(\mathcal{P}f)|_{c+\Omega_c}$ is the poisson extension of $f|_{c+S_c}$.

Dixmier trace formula

Let $h(z)$ be the harmonic extension of $\overline{p}q$. Then we have that for all $e \in S_1$

$$h(e) = \int_{S_e} (\overline{p}q)(e+c)d\sigma(c) = (p_e|q_e)_{S_e}$$

where S_e is the Shilov boundary of D_e, and $f_e(c) := f(e+c)$.

Theorem 2

$$tr_\omega[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] = \frac{1}{n!(2\pi)^n} \int_{S_1} \prod_j \mathcal{L}(\overline{\partial_b} \widehat{f}_j, \partial_b \widehat{g}_j) \eta \wedge (d\eta)^{n-1}$$

involving the boundary Poisson bracket of the harmonic extension $\widehat{f}_i, \widehat{g}_i$ restricted to S_1.

Thank you