Curvature properties of Cowen-Douglas classes and of Forelli-Rudin constructions.

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Outline

- Prove general positivity properties of curvatures of of C-D operators. Producing systematically examples on curvature inequalities vs factorization of reproducing kernels
- Compute Ricci curvature of infinite dimensional vector bundle in term of Toeplitz and Hankel operators. Trace/Dixmier trace formulas

Ω bounded domain in $\mathbb{C}^m$.

$B_n(\Omega)$:

Commuting tupe $T = (T_1, T_2, \cdots, T_m)$

$$\forall w \in \Omega, Ran(T - w) closed, \dim Ker(T - w) = n$$

$$\text{Span}\{Ker(T - w), w \in \Omega\}$$

is dense.

Vector bundle

$$E_T \subset \Omega \times H, H \supset E_w := Ker(T - w) \to w \in \Omega$$

The metric on H defines also a metric on E_T.

Curvature $R(X, Y) : E_w \mapsto E_w$.

$$R(X, Y)u = [\mathcal{D}_X, \bar{\partial}_Y]u = -\bar{\partial}_Y \mathcal{D}_X u$$

when acting on holomorphic sections, $\mathcal{D}_X$, Chern connection.

Cowen-Douglas theory. Further questions

Cowen-Douglas, ’77: Curvature R determines the unitary equivalence of T.

Question: Further correspondence between Θ and specific properties of T. (Contractivity, Subnormal ect.)

Biswas, Keshari, Misra, ’12: Contractivity/divisibility vs positivity.

Forelli-Rudin construction, '74

- $w(x)$ smooth weight function on Ω. $F \subset \mathbb{C}^m$ domain.
- $$\mathcal{X} = \{(t, z) \in \Omega \times \mathbb{C}^m; w(t)^{-\frac{1}{2}} z \in F\},$$

  viewed as a fiber bundle

  $$\mathcal{X}_t := \{z \in \mathbb{C}^m; w(t)^{-\frac{1}{2}} z \in f\} \mapsto t \in \omega$$
- Forelli-Rudin, '74; Ligocka, '89; Engliš, Englis-Zhang, '07: Computations of reproducing kernels on $\mathcal{X}$ in terms of bergman kernel on ω and f for some concrete ω and f.

Infinite dim. vector bundles. Generalization of F-R construction. Family of Bergman spaces.

- $\mathcal{X} \to \Omega$ holomorphic fibration of pseudo-convex domains, $\mathcal{X}_t \to t$. (For example: $\mathcal{X} = \Omega \times F$.
- ϕ, pluri-subharmonic function on $\mathcal{X}$, $\partial_i \bar{\partial}_j \phi$ positive definite.
- $$L_a^2(\mathcal{X}_t, e^{-\phi^t}) \mapsto t \in \Omega$$

  $\phi^t = \phi|_{\mathcal{X}_t}$, restriction of ϕ on $\mathcal{X}_t$

Positivity

1. Hermitian vector bundle (E, h) over some complex manifold T, $R(X, Y)$ curvature.
2. Griffith positive: nonzero vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$,

   $$\sum R_{i\bar{j}\alpha\bar{\beta}} u^i \bar{u}^j v^\alpha \bar{v}^\beta > 0,$$

   i.e.,

   $$\Theta(u \otimes v, u \otimes v) = g(R(u, \bar{u})(v), v) > 0$$

   $v \neq 0$ local holomorphic section u non-zero holomorphic tangent vector field.
3. Nakano positive if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

   $$\sum R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0,$$

   i.e., the associated sesquilinear form Θ is a positive definite Hermitian form.

Relations between the notions

1. Nakano positivity $\Longrightarrow$ Griffith positivity.
2. If E is Griffith positive then its dual E^* is Griffith negative.
3. Griffith positive then Ricci curvature is positive:

$$Ricci(X, X) > 0 \quad \forall X \neq 0$$

4. E line bundle, all notions agree.

Example

B, unit ball in $\mathbb{C}^n$ (of row vectors), equipped with the Bergman metric.

$$R(X, Y)Z = -(XY^*Z + ZY^*X),$$

Nagano negative.

Theorem 1. Positivity of Curvature of C-D operators I

(1) The curvature for Cowen-Douglas is Nagano seminegative. Griffith negative. In particular the Ricci curvature is always negative.

(2) Let (H, K) be a Hilbert space of V-vector holomorphic functions on a domain Ω. If all the constant functions are in H. Then the curvature is Nagano negative.

Theorem 1. Positivity of Curvature of C-D operators II

(3) Consider the infinite flag bundle (increasing vector bundles)

$$H \supset \cdots Ker(T-w)^2 \supset Ker(T-w) \mapsto w \in \Omega$$

Projections onto the complementary subspaces

$$0 \leq \cdots \leq \pi_2 \leq \pi_1 \leq I$$

$u(w)$ local holomorphic section of $\mathrm{Ker}(T-w)^n$

$$(R^n(X,X)u,u)(w_0) = -\sum_{j=1}^{\infty} \|\pi_{n+j}\mathcal{D}_X^{n+j}u\|^2(w_0)$$

Generalizes recent results of Biswas, Keshari, Misra.

Applications

Let $\Omega \subset \mathbb{C}$.

- Consider the problem of divisibility: K and L two reproducing kernel on Ω. When can we write

$$K = LQ$$

where Q is also a reproducing kernel.

- Reformulation: Given $c =$curvature of L positive real analytic function on Ω. Convex sets of functions

$$\Phi = \{\phi, \text{a subharmonic function on } \Omega, \partial_z \bar{\partial}_z \phi \geq c\}$$

and

$$\mathcal{K} = \{K = K(z,w) = e^{-\phi(z,w)}, \text{positive definite}, \text{curvature of } K \geq$$

- Produce systematically a large class of counter-examples with curvatures $R_1 \leq R_2$ and K_1 not divisible by K_2.

Infinite dim. vec. bundles. Berndtsson' 05, proving Griffith conjecture

Inf. dim. vec bundle

$$L_a^2(\mathcal{X}_t, e^{-\phi^t}) \mapsto t \in \Omega$$

introduced above.

Theorem 0.1

(Berndtsson) Suppose ϕ plurisubharmonic and $\mathcal{X}$ pseudo-convex. Then this bundle has its curvature Nagano positive.

Question: Further curvature properties. (Ricci curvature, Ricci metric of Ricci curvature etc)

Toeplitz and Hankel

D bounded domain in $\mathbb{C}^m$, Bergman operator $L^2_a(D, e^{-\phi}) \subset L^2(D, e^{-\phi})$, f function Ω, Toeplitz and Hankel operators on $L^2_a(D, e^{-\phi})$,

$$T_f : L^2_a \to L_a, \quad H_f : L^2_a \to L_a^{\perp}, \quad T_f = PM_fP, T_f = (I - P)M_fP.$$

Theorem 2. Trace of Curvature minus Toeplitz

Let $\mathcal{X}$ be a domain fibered over Ω, $\mathcal{X}_t \mapsto t$, $\mathcal{X}$, Ω, and $\mathcal{X}_t$ being pseudo-convex. Let ϕ be a plurisubharmonic function on Ω, smooth on the closure, and ϕ^t its restriction on $\mathcal{X}_t$. (1) Suppose $dim\mathcal{X}_t = 1$, i.e. one-dimensional fiber. Then the operator

$$\Theta(X, X) - T_{\partial_X \partial_{\bar{X}} \phi}$$

is trace class on $L_a^2(\Omega_t, e^{-\phi_t})$ and

$$\mathrm{tr}(\Theta(X, X) - T_{\partial_X \partial_{\bar{X}} \phi}) = -\|H_{\partial_X \phi}\|^2$$

where H_f is the Hankel operator on $L_a^2(\Omega, e^{-\phi_t})$.

(2) $dim \mathcal{X}_t > 1$. The operator

$$\Theta(X, X) - T_{\partial_X \partial_{\bar{X}} \phi}$$

is of Dixmier class $\mathcal{L}^{n,\infty}$ and

$$\mathrm{tr}_\omega(\Theta(X, X) - T_{\partial_X \partial_{\bar{X}} \phi})^n = -\int_{\partial_X \phi} J(\partial_X \phi)$$

can be computed using the boundary CR-Poisson bracket (Englis-Zhang, '10).

Bounded Symmetric Domains

1. F irreducible BSD of rank r, $0 \in F$, S Shilov boundary, $H^2(S)$ Hardy space. K isotropic subgroup of 0 in $Aut_0(F)$.
2. $H^2(S)$ Hardy space. Irreducible decomposition under K

$$H^2(S) = \sum_{\mathbf{m}: m_1 \geq m_2 \geq \cdots m_r \geq 0} P_{\mathbf{m}}$$

3. Example:

$$F = \{Z \in M_{r,r+b}(\mathbb{C}); Z * Z < I\}$$

Bergman, Hardy reproducing kernel

$$\det(I - W^* Z)^{-\nu}, \nu = r + b, \quad 2r + b.$$

Flag of subspace of Hardy space

Let

$$H_1 = \sum_{\mathbf{m}} P_{(m,0,\cdots,0)} \subset H_2 = \sum_{\mathbf{m}} P_{(m_1,m_2,0,\cdots,0)} \subset \cdots$$

Upmeier: $H^2(S_1)$ can be realized as Hardy space on a pseudo-convex manifold.

Result: Computation of Curvature tensor in terms of and Toeplitz operators consider by H. Upmeier ’84.
Dixmier trace properties of Toeplitz operators on H_1 studied by K. Wang and H. Upmeier ’15.