An alternate Toeplitz Corona Theorem

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BIRS

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The classical corona theorem

- In 1941, Kakutani asked if there was a corona in the maximal ideal space $\triangle$ of $H^{\infty}(\mathbb{D})$, i.e whether or not the disk $\mathbb{D}$ was dense in $\triangle$.

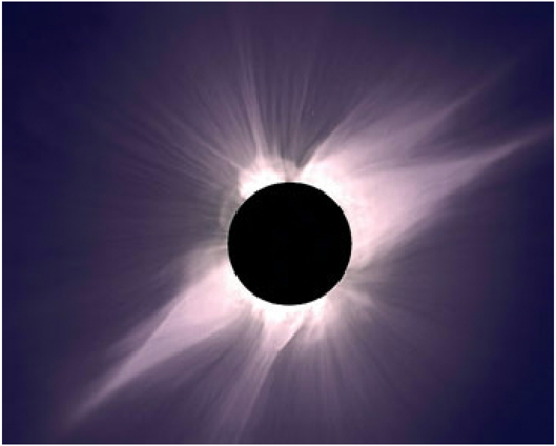
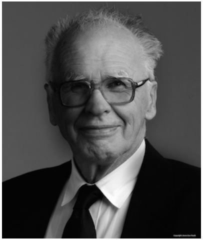


Figure: The sun's corona

Carleson's Corona Theorem

- In 1962 Lennart Carleson



showed that if $\varphi = \left\{\varphi_j\right\}_{j=1}^N \in \oplus^N H^\infty(\mathbb{D})$ satisfies

$$|\varphi(z)| = \sqrt{\sum_{j=1}^N \left|\varphi_j(z)\right|^2} \geq \delta > 0, \qquad z \in \mathbb{D}, \tag{1}$$

then there is $f = \{f_j\}_{j=1}^N$ in $\oplus^N H^\infty(\mathbb{D})$ with

$$f(z) \cdot \varphi(z) = \sum_{j=1}^N f_j(z)\,\varphi_j(z) = 1, \qquad z \in \mathbb{D}. \tag{2}$$

Bezout's equation

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$$f \cdot \varphi = 1 \text{ where } f, \varphi \in H^{\infty}(\mathbb{D}), \tag{3}$$

can be interpreted as exhibiting a bounded holomorphic inverse $f = (f_1, ..., f_N)$ for the bounded holomorphic vector function $\varphi = (\varphi_1, ..., \varphi_N)$ relative to the dot product.

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- Bezout's equation is equivalent to the weak * density of $\mathbb{D}$ (point evaluations) in the maximal ideal space $\triangle$ of H^{∞}, hence to the absence of a 'corona'.
- Despite intense efforts, the corona theorem remains open for $H^{\infty}(\mathbb{B}_n)$ when $n > 1$, the bounded analytic functions on the unit ball in $\mathbb{C}^n$.

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- The dot product $\cdot$ has been replaced in (3) with operator composition of bounded operators from one Hilbert space to another (Arveson, Schubert, Douglas, Helton), for example: φ is an $n \times m$ matrix with $m \leq n$ that is left invertible, equivalently $\varphi^* \varphi \succcurlyeq \delta^2 > 0$. One can take $m < \infty$ and $n = \infty$, but counterexamples (Treil 1989) show that $m = n = \infty$ fails in general, leading to the *Operator Corona Problem*.

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- The algebra $H^\infty(\mathbb{D})$ has been replaced with more general algebra function spaces, and in the case of a multiplier algebra M_H of a Hilbert space H, the lower bound (1) has been strengthened to the operator lower bound $T_\varphi T_\varphi^* \geq \delta^2$.

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- The target 1 has been replaced with more general $\psi \in H^\infty(\mathbb{D})$ and (1) weakened accordingly, leading to the *Ideal Problem*. The necessary condition $|\varphi| \gtrsim |\psi|$ is not sufficient in general (Rao), raising the question of for which h we have that the inequality $h(|\varphi|) \gtrsim |\psi|$ is sufficient for $f \cdot \varphi = \psi$.

Corona theorems in higher complex dimension

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2. Quite the opposite is true in higher dimensions. There are some counterexamples of domains of holomorphy Ω in $\mathbb{C}^n$ for which the $H^\infty(\Omega)$ corona theorem is known to fail, but there are **no** examples of domains Ω in $\mathbb{C}^n$ for which the $H^\infty(\Omega)$ corona theorem is known to hold (Costea, Sawyer and Wick 2011 have shown that the $M_{H_n^2}$ corona theorem holds where H_n^2 is the Drury-Arveson Hardy space on the ball).

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3. Key differences in higher dimensions are the existence of 'tangential' analytic disks near the boundary, and the complexity of zero sets.

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- This suggests yet another approach to Carleson's Corona Theorem in the disk.
- We prove a corona theorem for kernel multiplier spaces in the disk.

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- the argument of Trent and Wick exploiting the properties of **outer functions** in the disk.

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- In many cases, one can exploit the fact that a 'nice' holomorphic function in the disk can be recovered from its boundary values on the circle via the Poisson integral, and hence it suffices for such a function to control the sup on the circle alone. This fact extends to higher dimension.

Duality and Carleson measures

- In the disk, the duality and Carleson measure sup control can be rephrased as $i\int_{\mathbb{T}} u\left(e^{i\theta}\right) k\left(e^{i\theta}\right) e^{i\theta} d\theta$ equals

$$\begin{aligned} & \int_{\partial\mathbb{D}} u(z)\, k(z)\, dz = \int_{\mathbb{D}} d\left(u(z)\, k(z)\, dz\right) \\ = & \int_{\mathbb{D}} \left(\partial + \overline{\partial}\right)\left(u(z)\, k(z)\, dz\right) = \int_{\mathbb{D}} \overline{\partial} u(z)\, k(z)\, d\overline{z} \wedge dz \\ = & \int_{\mathbb{D}} k(z)\, d\mu(z), \qquad \text{for all } \overline{\partial}\left[k(z)\, dz\right] = 0. \end{aligned}$$

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- In higher dimensions this becomes

$$\int_{\partial\mathbb{B}_n} u \wedge k = \int_{\mathbb{B}_n} \mu \wedge k,$$

for all $k = \sum_{j=1}^{n} k_j(z)\ \widehat{d\overline{z}}^{(j)} \wedge dz \in C^{\infty}_{n,n-1}(\mathbb{B}_n)$ such that $\overline{\partial} k = 0$ in a neighbourhood of $\overline{\mathbb{B}_n}$, i.e. $\frac{\partial k_i}{\partial \overline{z}_j}(z) = \frac{\partial k_j}{\partial \overline{z}_i}(z)$ for $i \neq j$.

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The Hopf map

- Parameterize $\partial\mathbb{B}_2 \setminus \{(z_1, z_2) : z_1 = 0\}$ by the Hopf variables $(u, \theta) \in \mathbb{C} \times [0, 2\pi)$ where $z_1 = \frac{e^{i\theta}}{\sqrt{1+|u|^2}}$ and $z_2 = \frac{ue^{i\theta}}{\sqrt{1+|u|^2}}$.

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- If $f(u) \in BMO(\mathbb{R}^2) \setminus L^\infty(\mathbb{R}^2)$ has compact support, then $\widetilde{f}(u, \theta) = f(u) \in BMO(\partial\mathbb{B}_2; d)$ and there is **no** decomposition of the form

$$\widetilde{f} = \varphi + \psi, \tag{4}$$

where $\varphi \in L^\infty(\partial\mathbb{B}_2)$ is bounded and $\psi \in H^2(\partial\mathbb{B}_2)$ is in the classical Hardy space.

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where $\varphi \in L^\infty(\partial\mathbb{B}_2)$ is bounded and $\psi \in H^2(\partial\mathbb{B}_2)$ is in the classical Hardy space.
- Indeed, integrating in θ over the Hopf fibres and using that $\psi \in H^2(\mathbb{B}_2)$ we contradict the assumption that $f \notin L^\infty(\mathbb{R}^2)$:

$$\begin{aligned} f(u) &= \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(u, \theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} [\varphi(u, \theta) + \psi(u, \theta)]\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(u, \theta)\, d\theta + \psi_u(0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(u, \theta)\, d\theta + \psi(0). \end{aligned}$$

The Varopolous counterexample

- Varopolous claims there is a $\overline{\partial}$-closed $(0,1)$-form Carleson measure μ on the unit ball $\mathbb{B}_2$ in $\mathbb{C}^2$ such that the d-bar boundary equation $\overline{\partial_b} u = \mu$ has **no** bounded solution $u \in L^\infty(\partial\mathbb{B}_2)$.

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- Assume this fails. Given $\widetilde{f} \in BMO(\partial\mathbb{B}_2)$, a theorem of Varopoulos gives a bounded function $w \in L^\infty(\partial\mathbb{B}_2)$ and a $\overline{\partial}$-closed $(0,1)$-form Carleson measure μ such that

$$\overline{\partial_b}\left(\widetilde{f} - w\right) = \mu, \ i.e. \int_{\partial\mathbb{B}_2}\left(\widetilde{f} - w\right) \wedge k = \int_{\mathbb{B}_2} \mu \wedge k,$$

for all $k \in C^\infty_{2,1}(\mathbb{B}_2)$ such that $\overline{\partial} k = 0$ in a neighbourhood of $\overline{\mathbb{B}_2}$.

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- By our assumption of failure, there is a bounded function $u \in L^\infty(\partial\mathbb{B}_2)$ such that

$$\overline{\partial_b} u = \mu.$$

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- So both w and u are bounded and $\overline{\partial_b}\left(\widetilde{f}-w-u\right)=0$,hence $\widetilde{f}-w-u\in H^2\left(\partial\mathbb{B}_2\right)$, hence
$$\widetilde{f}\in L^\infty\left(\partial\mathbb{B}_2\right)+H^2\left(\partial\mathbb{B}_2\right),$$
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- *So it might be possible to solve the d-bar equation with a bounded solution for corona data measures*;
- *Or it might be necessary to use a soft approach like the Toeplitz Corona Theorem.*

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The multiplier algebra approach

Schubert, Arveson, Douglas and Helton

- The multiplier algebra $M_{H^2(\mathbb{D})}$ of the classical Hardy space $H^2(\mathbb{D})$ is $H^\infty(\mathbb{D})$. If Bezout's equation (3) can be solved in $H^\infty(\mathbb{D})$, then the following baby version for $H^2(\mathbb{D})$ holds: for every $h \in H^2(\mathbb{D})$ there are $f_1, \ldots, f_N \in H^2(\mathbb{D})$ such that the baby Bezout equation holds:

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- To obtain (5) from Bezout (3), simply multiply $f \cdot \varphi = 1$ by h to get $(fh) \cdot \varphi = h$, and note that $fh \in H^2(\mathbb{D})$ since $f \in M_{H^2(\mathbb{D})}$.

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- The Toeplitz corona theorem, which requires the reproducing kernel $\frac{\sqrt{1-|w|^2}}{1-\overline{w}z}$ of $H^2(\mathbb{D})$ to be a complete Pick kernel, shows that if the baby corona theorem holds for $H^2(\mathbb{D})$ with bounds, then the corona theorem holds for $H^\infty(\mathbb{D})$ with the **same** bounds, thus giving the necessary **sup control**.

- The multiplier algebra $M_{H^2(\mathbb{D})}$ of the classical Hardy space $H^2(\mathbb{D})$ is $H^\infty(\mathbb{D})$. If Bezout's equation (3) can be solved in $H^\infty(\mathbb{D})$, then the following baby version for $H^2(\mathbb{D})$ holds: for every $h \in H^2(\mathbb{D})$ there are $f_1, \ldots, f_N \in H^2(\mathbb{D})$ such that the baby Bezout equation holds:

$$f \cdot \varphi = h \text{ in } \mathbb{D}. \tag{5}$$

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- In the baby corona theorem, **sup control** is no longer needed, and it turns out that the baby theorem can be generalized to higher dimensions - but not the Toeplitz corona theorem!

Failure of the complete Pick property

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- For example, the kernel $k(z,w)=\frac{1}{1-\overline{w}\cdot z}$ on the ball $\mathbb{B}_n$ is a complete Pick kernel, but the Cauchy kernel $\left(\frac{1}{1-\overline{w}\cdot z}\right)^n$ on the ball is not. The Drury-Arveson Hardy space H_n^2 on the ball has reproducing kernel $\frac{1}{1-\overline{w}\cdot z}$, and the classical Hardy space $H^2(\mathbb{B}_n)$ on the ball has reproducing kernel $\left(\frac{1}{1-\overline{w}\cdot z}\right)^n$. The two spaces coincide only when $n=1$.

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- The Toeplitz corona theorem asserts that if a Hilbert function space has a complete Pick kernel, then the baby corona theorem for a Hilbert function space H is equivalent to the corona theorem for its multiplier algebra M_H, and moreover with identical bounds. While the baby corona theorem is known for the classical Hardy space $H^2(\mathbb{B}_n)$, the Toeplitz corona theorem doesn't apply when $n>1$ to give the corona theorem for its multiplier algebra $H^\infty(\mathbb{B}_n)$.

The outer function approach

Agler-McCarthy, Amar, Trent-Wick

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- Trent and Wick then further reduced matters to checking weights whose densities are the modulus squared of nonvanishing H^∞ functions whose boundary values have bounded reciprocals (but the reciprocals may not be well-behaved in the ball or polydisc).
- In dimension $n = 1$ the existence and properties of outer functions establishes the weighted estimates of Trent-Wick, giving the necessary **sup control**, and thus they obtain yet another proof of the corona theorem in the disk.

Nonexistence of good outer functions

- If $\ln \psi \in L^1(\mathbb{T})$, the outer function

$$h(z) \equiv \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\ln\psi\left(e^{it}\right)dt\right\}, \qquad z \in \mathbb{D},$$

satisfies $\left|h^*\left(e^{it}\right)\right| = \psi\left(e^{it}\right)$ for almost every $0 \le t < 2\pi$. Moreover if $\ln\psi$ is bounded, then $h, \frac{1}{h} \in H^\infty(\mathbb{D})$.

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- In the polydisc, Rudin proved much earlier that if f is a continuous positive function on $\mathbb{T}^n$, then there is a positive singular measure σ on $\mathbb{T}^n$ such that the Poisson integral $\mathbb{P}(f-\sigma)(z)$ of the real measure $f-\sigma$ is the real part of a holomorphic function. Alexandrov and Løw proved a version of this on the ball and used it to solve the famous inner function problem in the ball.

Quick review of Hilbert function spaces

- A Hilbert space $\mathcal{H}$ is a *Hilbert function space* (aka a reproducing kernel Hilbert space) on a set Ω if the elements of $\mathcal{H}$ are complex-valued functions f on Ω with the usual vector space structure, such that each point evaluation on $\mathcal{H}$ is a nonzero continuous linear functional.

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- There is then a unique element $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}} \text{ for all } x \in \Omega.$$

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- The function $k(y, x) \equiv \langle k_x, k_y \rangle_{\mathcal{H}} = k_x(y)$ is self-adjoint and positive semidefinite, written $k \succcurlyeq 0$. We call such a function k a kernel function.

Quick review of Hilbert function spaces

- Given any kernel function k on $\Omega \times \Omega$, define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_k}$ on finite linear combinations $\sum_{i=1}^{N} \xi_i k_{x_i}$ of the functions $k_{x_i}(\zeta) = k(\zeta, x_i)$, $\zeta \in \Omega$, by

$$\left\langle \sum_{i=1}^{N} \xi_i k_{x_i}, \sum_{j=1}^{N} \eta_j k_{x_j} \right\rangle_{\mathcal{H}_k} = \sum_{i,j=1}^{N} \xi_i \overline{\eta_j} k(x_j, x_i), \tag{6}$$

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- The Hilbert space $\mathcal{H}_k$ has kernel k, and if $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert function spaces on Ω that have the same kernel function k, then there is an isometry from $\mathcal{H}$ onto $\mathcal{H}'$ that preserves the kernel functions k_x, $x \in \Omega$.

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- The shifted space $\mathcal{H}^a$ is the Hilbert space with inner product $\langle f, g \rangle_{\mathcal{H}^a} = \left\langle \widetilde{k}_a f, \widetilde{k}_a g \right\rangle_{\mathcal{H}}$, where $\widetilde{k}_a = \frac{k_a}{\|k_a\|}$ is the normalized kernel.

Quick review of Hilbert function spaces

Multiplier and kernel multiplier algebras

- The Banach algebra $M_{\mathcal{H}}$ of (pointwise) multipliers of $\mathcal{H}$ consists of all functions φ on Ω for which

$$\|\varphi\|_{M_{\mathcal{H}}} \equiv \|\mathcal{M}_{\varphi}\|_{\mathcal{H}\to\mathcal{H}} \equiv \sup_{f\in\mathcal{H}:\ f\neq 0} \frac{\|\varphi f\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}} < \infty.$$

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- We have $M_{\mathcal{H}} \hookrightarrow K_{\mathcal{H}} \hookrightarrow \mathcal{H} \cap L^{\infty}$. $K_{\mathcal{H}}$ is an algebra for all of the Hardy-Sobolev spaces in higher dimension.

The convex Poisson condition

Definition

Let $\mathcal{H}$ be a Hilbert function space on a set Ω with nonvanishing kernel, and let $\mathcal{H}^a$ be the shifted Hilbert space for $a \in \Omega$. We say that a vector $\varphi \in \oplus_{\ell=1}^N L^\infty(\Omega)$ satisfies the $\mathcal{H}-$*convex Poisson condition* with positive constant C if for every finite collection of points $\mathbf{a} = (a_1, \ldots, a_M) \in \Omega^M$ and every collection of nonnegative numbers $\boldsymbol{\theta} = \{\theta_m\}_{m=0}^M$ summing to $1 = \sum_{m=0}^M \theta_m$, there is a vector $g^{\mathbf{a},\boldsymbol{\theta}} \in \oplus_{\ell=1}^N \mathcal{H}$ satisfying

$$\begin{aligned} \varphi(z) \cdot g^{\mathbf{a},\boldsymbol{\theta}}(z) &= 1, \qquad z \in \Omega, \\ \left\| g^{\mathbf{a},\boldsymbol{\theta}} \right\|^2_{\oplus_{\ell=1}^N \mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} &= \theta_0 \left\| g^{\mathbf{a},\boldsymbol{\theta}} \right\|^2_{\oplus_{\ell=1}^N \mathcal{H}} + \sum_{m=1}^M \theta_m \left\| g^{\mathbf{a},\boldsymbol{\theta}} \right\|^2_{\oplus_{\ell=1}^N \mathcal{H}^{a_m}} \leq C^2 . \end{aligned} \tag{7}$$

We denote the smallest such constant C by $\|\varphi\|_{cPc}$.

Convex Poisson Property

We obtain an analogue of the Toeplitz Corona Theorem for the kernel multiplier space $K_{\mathcal{H}}$ when it is an algebra. The role of the Baby Corona Property for $\mathcal{H}$ will be played by the following property.

Definition

Let $\mathcal{H}$ be a Hilbert function space with kernel k on a set Ω, and let $c, C > 0$. We say that the space $\mathcal{H}$ has the *Convex Poisson Property* with positive constants c, C if for all vectors $\varphi \in \oplus^N K_{\mathcal{H}}$ satisfying $\|\varphi\|_{\oplus^N K_{\mathcal{H}}} \leq 1$ and

$$|\varphi_1(z)|^2 + \cdots + |\varphi_N(z)|^2 \geq c^2 > 0, \qquad z \in \Omega, \tag{8}$$

the vector φ satisfies the $\mathcal{H}-$convex Poisson condition with constant C.

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   3. Note that we make no assumptions regarding the size of the norms of the multipliers k_x and $\frac{1}{k_x}$ in this definition. All the Hardy-Sobolev spaces on the ball are multiplier stable, as well as the Bergman and Hardy spaces on strictly pseudoconvex domains with C^2 boundary.

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2. A consequence of the multiplier stable assumption is the $\mathcal{H}$-Poisson reproducing formula. Suppose $\mathcal{H}$ is a Hilbert function space on a set Ω with nonvanishing kernel and containing the constant functions. Suppose furthermore that $k_x \in M_{\mathcal{H}}$ for all $x \in \Omega$. Then for each $a \in \Omega$ we have

$$f(a) = \langle f, 1 \rangle_{\mathcal{H}^a}, \qquad f \in \mathcal{H}(\Omega), \ a \in \Omega. \tag{9}$$

Montel property

Definition

Let Ω be a topological space. A Hilbert function space $\mathcal{H}$ of continuous functions on Ω is said to be have the *Montel property* if there is a dense subset S of Ω with the property that for every sequence $\{f_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$, there are a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and a function g in the unit ball of $\mathcal{H}$, such that

$$\lim_{k\to\infty} f_{n_k}(x) = g(x), \qquad x \in S.$$

Invertible Multiplier Property

Definition

Let $\mathcal{H} = \mathcal{H}_k$ be a multiplier stable Hilbert function space on a set Ω with reproducing kernel k, and containing the constant functions. We say that the kernel k has the *Invertible Multiplier Property* if for every $(\mathbf{a}, \boldsymbol{\theta}) \in \Omega^M \times \Sigma_M(0)$, there is a normalized invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} \in M_{\mathcal{H}}$ such that

$$\langle f, g \rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \left\langle \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} f, \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} g \right\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}. \tag{10}$$

- The normalized invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ in (10) is uniquely determined.

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- The normalized invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ in (10) is uniquely determined.
- This property fails for the classical Hardy spaces H^2 on the ball and polydisc in dimension greater than 1.

An alternate Toeplitz corona theorem

Sawyer-Wick - there is different alternate theorem by Douglas and Sarkar

Theorem

Suppose that $\mathcal{H}$ is a multiplier stable Hilbert function space of continuous functions on Ω that contains the constant functions, and enjoys the Montel property. Suppose further that the space of kernel multipliers $K_{\mathcal{H}}$ is an algebra.

1. *Then $K_{\mathcal{H}}$, with the direct sum $\bigoplus^N K_{\mathcal{H}}$ normed by $\|\cdot\|_{\bigoplus^N K_{\mathcal{H}}}$, satisfies the Corona Property with positive constants c, C if and only if $\mathcal{H}$ satisfies the Convex Poisson Property with positive constants c, C.*
2. *Suppose in addition that $\mathcal{H}$ satisfies the Invertible Multiplier Property and that $M_{\mathcal{H}} = K_{\mathcal{H}}$ isometrically. Equip the direct sum $\bigoplus^N M_{\mathcal{H}}$ with the norm $\|\cdot\|_{\bigoplus^N M_{\mathcal{H}}}$.*
   1. *Then $\mathcal{H}$ satisfies the Baby Corona Property with constants c, C if $M_{\mathcal{H}}$ satisfies the Corona Property with the constants c, C.*
   2. *Conversely, $M_{\mathcal{H}}$ satisfies the Corona Property with constants $c, C\sqrt{N}$ if $\mathcal{H}$ satisfies the Baby Corona Property with constants c, C.*

- The formula $k_a k_z = \frac{1}{\overline{z}-\overline{a}}\left(\overline{z}k_z - \overline{a}k_a\right)$ for the Szegö kernel shows that the operator $T^{\mathbf{a},\boldsymbol{\theta}}$ defined by $\langle f, g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \left\langle f, T^{\mathbf{a},\boldsymbol{\theta}} g\right\rangle_{\mathcal{H}}$ is given by

$$\begin{aligned} T^{\mathbf{a},\boldsymbol{\theta}} &= \sum_{m=0}^{M} \theta_m \left(1 - |a_m|^2\right) M_{k_{a_m}}^* M_{k_{a_m}} \\ &= \sum_{m=0}^{M} \theta_m |a_m|^2 \left[k_{a_m} \otimes k_{a_m}\right] + \sum_{m=0}^{M} \theta_m \left(1 - |a_m|^2\right) M_{k_{a_m}} M_{k_{a_m}}^* \ . \end{aligned}$$

A seventh approach to Carleson's Corona Theorem

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- The reproducing kernel $k^{\mathbf{a},\boldsymbol{\theta}}(z,w)$ for the space $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}$ is given by

$$k_w^{\mathbf{a},\boldsymbol{\theta}}(z) = \left(T^{\mathbf{a},\boldsymbol{\theta}}\right)^{-1} k_w(z) .$$

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- The convex Poisson property for the Szegö kernel can then be verified by hand in some simple cases of $\mathbf{a},\boldsymbol{\theta}$.

The corona theorem for kernel multiplier spaces

Sawyer-Wick

The Corona theorem holds for the one-dimensional algebras of kernel multipliers $K_{\mathcal{H}}$ on the Hilbert spaces $\mathcal{H} = B_2^\sigma(\mathbb{D})$ for $0 < \sigma \leq \frac{1}{2}$ where

$$B_2^\sigma(\mathbb{D}) \equiv \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} \left| f^{(1-\sigma)}(z) \right|^2 dA(z) < \infty \right\}.$$

Theorem

Let $N \geq 2$, $0 < \sigma \leq \frac{1}{2}$ and suppose that $\varphi_1, \ldots, \varphi_N \in K_{B_2^\sigma(\mathbb{D})}$ satisfy

$$1 \geq \max\left\{ |\varphi_1(z)|^2, \ldots, |\varphi_N(z)|^2 \right\} \geq c > 0, \qquad z \in \mathbb{D}.$$

Then there are a positive constant C and $f_1, \ldots, f_N \in K_{B_2^\sigma(\mathbb{D})}$ satisfying

$$\begin{aligned} \max\left\{ |f_1(z)|^2, \ldots, |f_N(z)|^2 \right\} &\leq C, \qquad z \in \Omega, \\ \varphi_1(z) f_1(z) + \cdots + \varphi_N(z) f_N(z) &= 1, \qquad z \in \Omega. \end{aligned}$$

Selected Open Problems

- Do the algebras $H^{\infty}(\mathbb{B}_n), H^{\infty}(\mathbb{D}^n)$ of bounded analytic functions on the ball and polydisc have a corona in their maximal ideal spaces? Obstacles include the lack of Blaschke products, the failure of the complete Pick property for $H^2(\mathbb{B}_n), H^2(\mathbb{D}^n)$, and the failure of the invertible multiplier property on the ball and polydisc.

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- Thanks!