Matrix Concomitants, Schemes, and Azumaya Algebras

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Outline

Three Objectives

- To show how noncommutative functions as described in "Foundations of Free Noncommutative Function Theory" by Dima Kaliuzhnyi-Verbovetskyi and Victor Vinnikov [Kaliuzhnyi-Verbovetskyi & Vinnikov, 2014], can be realized and studied as cross sections of holomorphic matrix bundles.
- To suggest how Arveson's theory of subalgebras of C^*-algebras may interact with the theory.
- To suggest roles that work in contemporary algebra and geometry may play in the theory.

Matrix Concomitants

- Throughout, G denotes $PGL(n, \mathbb{C})$, viewed as the automorphisms of $M_n(\mathbb{C})$; K will denote $PU(n, \mathbb{C})$, viewed as the *-automorphisms of $M_n(\mathbb{C})$. We will abuse notation and write $s \cdot a$, $a \cdot s$ or $s^{-1}as$, for $s \in G$ and $a \in M_n(\mathbb{C})$ depending on what seems most natural.

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- Suppose H is an algebraic group acting algebraically on an algebraic variety X and suppose $\rho : H\to G$ is an algebraic representation H in G. A matrix concomitant on X is simply a rational map $F : X\to M_n(\mathbb{C})$ such that $F(xh)=\rho(h)\cdot F(x)$.

Matrix Concomitants

- Clearly, the matrix concommitants on X form a subalgebra $M_n(\mathbb{C}(X))$, where $\mathbb{C}(X)$ is the function field of X.

The Free Algebra

- Let $\mathbb{F}_d^+$ be the free semigroup on d symbols $\{1, 2, \cdots, d\}$. Elements are words $w = i_1 i_2 \cdots i_k$.
- The free algebra in d variables is $\mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle := \mathbb{C}^{(\mathbb{F}_d^+)}$ - the semigroup algebra of $\mathbb{F}_d^+$.
- $(a+b)(w) := a(w) + b(w)$, $a * b(w) := \sum_{w=uv} a(u)b(v)$.
- The "indeterminate" X_i is the function $X_i(w) = 1$, if $w = i$, and $X_i(w) = 0$ otherwise.

The n-dimensional representations of the Free Algebra

- Each algebra homomorphism $\pi : \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle \to M_n(\mathbb{C})$ is given by a d-tuple $\mathfrak{z} := (Z_1, Z_2, \cdots, Z_d)$, $Z_i \in M_n(\mathbb{C})$.
- $\pi(X_i) = Z_i$.
- Write $\pi_{\mathfrak{z}} := \pi$, and for $a \in \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$, write $\pi_{\mathfrak{z}}(a) = \widehat{a}_n(\mathfrak{z}) = \sum_{w \in \mathbb{F}_d^+} a(w) Z^w$, where $Z^w := Z_{i_1} Z_{i_2} \cdots Z_{i_k}$.
- $\mathbb{G}_0(d, n) := \{\widehat{a}_n(\cdot) \mid a \in \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle\}$ - the algebra of d generic $n \times n$ matrices.
- $\mathbb{G}_0(d, n)$ is a very small subalgebra of $M_n(\mathbb{C}[M_n^d]) := M_n(\mathbb{C}[z_{ij}(k), 1 \leq i, j \leq n, 1 \leq k \leq d])$ - the M_n-valued polynomial functions on $M_n(\mathbb{C})^d$, viewed as $\mathbb{C}^{dn^2}$.

Theorem

[Kaliuzhnyi-Verbovetskyi & Vinnikov, 2014, Thm 6.1] Given $\{f_n\}_{n\geq 1}$, *with* $f_n \in M_n(\mathbb{C}[M_n^d])$ *and* $\sup_n \deg f_n < \infty$, *then there is an* $a \in \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$ *such that* $f_n = \widehat{a}_n$, $n \geq 1$, *iff*

1. $f_{k+l}(\begin{bmatrix} \mathfrak{z} & 0 \\ 0 & \mathfrak{w} \end{bmatrix}) = \begin{bmatrix} f_k(\mathfrak{z}) & 0 \\ 0 & f_l(\mathfrak{w}) \end{bmatrix}$, $\mathfrak{z} \in M_k(\mathbb{C})^d$, *and* $\mathfrak{w} \in M_l(\mathbb{C})^d$.
2. $f_n(s^{-1}\mathfrak{z}s) = s^{-1}f_n(\mathfrak{z})s$, $\mathfrak{z} \in M_n(\mathbb{C})^d$, $s \in GL_n(\mathbb{C})$.

- Thus each f_n is a polynomial matrix concomitant on M_n^d.

Definition

$Hol(M_n^d, M_n)^G :=$ all holomorphic matrix concomitants.

FIRST GOAL

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- Would like a sheaf, more fully, a scheme of holomorphic concomitants.
- What does that mean? See [Luminet, 1997].

Perspective

- Would like to write $Hol(M_n^d, M_n)^G$ as "nice" functions on the space of similarity classes of irreducible n-dimensional representations of $\mathbb{C}\langle X_1, X_2, \cdots, X_d \rangle$.
- A representation $\pi_{\mathfrak{z}} : \mathbb{C}\langle X_1, X_2, \cdots, X_d \rangle \to M_n$ is irreducible if its image has no invariant subspaces,

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- A representation $\pi_{\mathfrak{z}} : \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle \to M_n$ is irreducible if its image has no invariant subspaces, equivalently $\pi_{\mathfrak{z}}$ is surjective.
- $\pi_{\mathfrak{z}}$ is similar to $\pi_{\mathfrak{w}}$ iff there is an $s \in G$ such that $s^{-1}\pi_{\mathfrak{z}}(a)s = \pi_{\mathfrak{w}}(a)$ for all $a \in \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$, if and only if $s^{-1}Z_i s = W_i$ for all i, i.e., $s^{-1}\pi_{\mathfrak{z}}(\cdot)s = \pi_{s^{-1}\mathfrak{z}s}(\cdot)$.

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- Can we replace M_n^d/G with a nice space?

Griesenauer, Muhly, and Solel

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- Think Jordan canonical form. The similarity orbit of a matrix is closed if and only if the matrix is diagonable!
- Can we replace M_n^d/G with a nice space?
- Yes! The categorical quotient $M_n^d//G$ - to be defined.

On Invariants

- $\mathbb{I}_0(d,n) := \{f \in \mathbb{C}[M_n^d] \mid f(s^{-1}\mathfrak{z}s) = f(\mathfrak{z}), s \in G, \mathfrak{z} \in M_n^d\}$.
- $\mathbb{I}_0(d,n)$ is finitely generated [Hilbert, 1890].
- $Q(d,n) := Spec(\mathbb{I}_0(d,n))$ - an abstract algebraic variety.
- The inclusion $\mathbb{I}_0(d,n) \to \mathbb{C}[M_n^d]$ induces a morphism $\pi : M_n^d \to Q(d,n)$. (A "concrete" picture of $Q(d,n)$ will be provided shortly.)

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- If $\mathbb{I}_0(d,n)$ is generated by $f_1, f_2, \cdots, f_p$, then π may be viewed as the map $\mathfrak{z} \to (f_1(\mathfrak{z}), \cdots, f_p(\mathfrak{z}))$ and $Q(d,n)$ may be viewed as its image in $\mathbb{C}^p$.

$2 \times 2 \times 2$ Example

Theorem

[Le Bruyn, 2008, Sect. 1.2] For $\mathfrak{z} = (Z_1, Z_2)$ *in* M_2^2, *let* $f_1(\mathfrak{z}) := tr(Z_1)$, $f_2(\mathfrak{z}) = tr(Z_2)$, $f_3(\mathfrak{z}) = \det(Z_1)$, $f_4(\mathfrak{z}) = \det(Z_2)$, *and* $f_5(\mathfrak{z}) = tr(Z_1 Z_2)$. *Then*

1. *The* f_i*'s generate* $\mathbb{I}_0(2,2)$.
2. *The* f_i*'s are algebraically independent.*
3. $\mathbb{I}_0(2,2) \simeq \mathbb{C}[\mathbb{C}^5]$, *so* $Q(2,2) \simeq \mathbb{C}^5$.

Artin-Procesi and the Categorical Quotient

- Write $\mathfrak{z}G$ for $\{s^{-1}\mathfrak{z}s \mid s \in G\}$, the similarity orbit of $\mathfrak{z}$.

Theorem

[Artin, 1969, 12.6] [Procesi, 1974, 4.1]

1. *$\mathfrak{z}G$ is closed iff the image of $\pi_{\mathfrak{z}}$ is a semisimple subalgebra of M_n; $\mathfrak{z}$ is called semisimple in this case.*
2. *For all $\mathfrak{z} \in M_n^d$, $\overline{\{s^{-1}\mathfrak{z}s \mid s \in G\}} := \overline{\mathfrak{z}G}$ contains a unique closed orbit, $\mathfrak{z}_{ss}G$, where $\mathfrak{z}_{ss}$ is the "semisimplification" of $\mathfrak{z}$.*
3. *If $\mathfrak{z}$ and $\mathfrak{w}$ are semisimple, then $\pi(\mathfrak{z}) = \pi(\mathfrak{w})$ if and only $\mathfrak{z}G = \mathfrak{w}G$. Thus $Q(d,n)$ may be viewed as parameterizing either the closed orbits or the orbit closures in M_n^d.*
4. *If $\rho : M_n^d \to X$ is any equivariant regular map into an algebraic variety, then ρ factors through π, i.e., there is a unique regular map $f : Q(d,n) \to X$ such that $\rho = f \circ \pi$, i.e., $Q(d,n)$ is the categorical quotient $M_n^d//G$.*

The Trace Algebra

Definition

The trace algebra $\mathbb{S}_0(d,n)$ is the subalgebra of $M_n(\mathbb{C}[M_n^d])$ generated by $\mathbb{G}_0(d,n)$ and $\mathbb{I}_0(d,n)$, where $\mathbb{I}_0(d,n)$ is viewed as $\mathbb{C}I_n$-valued functions.

Theorem

[Procesi, 1976, Thms 2.1 and 3.4]

1. $\mathbb{I}_0(d,n)$ *is generated by* $\mathfrak{z} \to tr(Z_{i_1}Z_{i_2}\cdots Z_{i_s})I_n$, $s \leq 2^n - 1$.
2. $\mathbb{S}_0(d,n) = M_n(\mathbb{C}[M_n^d])^G$ *- the polynomial matrix concomitants.*
3. $\mathfrak{Z}\mathbb{S}_0(d,n) = \mathbb{I}_0(d,n)$.
4. $\mathbb{S}_0(d,n)$ *is spanned as a module over* $\mathbb{I}_0(d,n)$ *by* $\mathfrak{z} \to Z_{i_1}Z_{i_2}\cdots Z_{i_s}$, $s \leq 2^n - 2$.

Irreducible points

Definition

$\mathscr{V}(d,n) := \{\mathfrak{z} \mid Z_1, Z_2, \cdots, Z_d \text{ generate } M_n(\mathbb{C})\}$ - the irreducible points of $M_n(\mathbb{C})^d$. $Q_0(d,n) := \pi(\mathscr{V}(d,n))$.

Fact

$\mathscr{V}(d,n)$ is open and Zariski dense in $M_n(\mathbb{C})^d$, i.e., $\mathscr{V}(d,n)^c$ is an algebraic variety.

Procesi's Fundamental Theorem

Theorem

[Procesi, 1974, Thm 5.10]

1. $Q_0(d,n)$ *is contained in the smooth points of* $Q(d,n)$.
2. $\mathscr{V}(d,n)$ *is the bundle space of a nontrivial, holomorphic, principal* G*-bundle*, $\mathfrak{P} = \mathfrak{P}(d,n)$, *over* $Q_0(d,n)$ *and* π *(restricted to* $\mathscr{V}(d,n)$*) is the bundle map; in particular, the action of* G *on* $\mathscr{V}(d,n)$ *is free and proper, and* π *identifies* $\mathscr{V}(d,n)/G$ *with* $Q_0(d,n)$.

- Thus the similarity classes of irreducible n-dimensional representations of $\mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$ are nicely parametrized by $Q_0(d,n)$.

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- Thus the similarity classes of irreducible n-dimensional representations of $\mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$ are nicely parametrized by $Q_0(d,n)$.
- The open locus of $\pi : M_n^d \to Q(d,n)$ is $\mathscr{V}(d,n)$, except when $n = d = 2$ (Le Bruyn).

Example: The $2 \times 2 \times 2$ Matrices (bis)

- $\mathscr{V}(2,2) = \{(Z_1, Z_2) \mid \det(Z_1 Z_2 - Z_2 Z_1) \neq 0\}$.
- $Q_0(2,2) = \{(z_1, z_2, \cdots, z_5) \mid z_5^2 - z_1 z_3 z_5 + z_1^2 z_4 + z_3^2 z_2 - 4 z_2 z_4 \neq 0\}$ [Le Bruyn, 2008, p. 16]
- $\mathbb{S}_0(2,2)$ is a free module over $\mathbb{I}_0(2,2)$, generated by

$$\mathfrak{z} \to \begin{cases} I_2 \\ Z_1 \\ Z_2 \\ Z_1 Z_2 \end{cases}.$$

- Every $F \in \mathbb{S}_0(2,2)$ can be written uniquely as $F(\mathfrak{z}) = \xi_1(\mathfrak{z}) + \xi_2(\mathfrak{z}) Z_1 + \xi_3(\mathfrak{z}) Z_2 + \xi_4(\mathfrak{z}) Z_1 Z_2$, $\xi_i \in \mathbb{I}_0(2,2)$, $\mathfrak{z} \in \mathscr{V}(2,2)$.

Cross Sections and Matrix Concomitants

- Let $\mathfrak{M}_n := \mathfrak{P}[M_n]$ be the M_n-bundle over $Q_0(d,n)$ associated to $\mathfrak{P}$. The bundle space of $\mathfrak{M}_n$ is $\mathscr{V}(d,n) \times_G M_n$, the quotient of $\mathscr{V}(d,n) \times M_n$ under the right action of G: $(\mathfrak{z}, m) \cdot s := (s^{-1}\mathfrak{z}s, s^{-1}ms)$. The bundle projection $\pi([\mathfrak{z}, m]) := \pi(\mathfrak{z})$.

Observation (Griesenauer, M and Solel)

$Hol(M_n^d, M_n)^G$ is isomorphic to the algebra of holomorphic cross sections of $\mathfrak{M}_n$, $\Gamma_h(Q_0(d,n), \mathfrak{M}_n)$, via the map $F \to s_F$, where $s_F(\mathfrak{z}G) = [\mathfrak{z}, F(\mathfrak{z})]$.

Proposal

The sheaf of holomorphic concomitants should be the sheaf $\underline{\Gamma_h(Q_0(d,n), \mathfrak{M}_n)}$ of holomorphic sections of $\mathfrak{M}_n$.

Getting a feel for $\mathfrak{P}$ and $\mathfrak{M}_n$

Definition

An *n*-variety is a *G*-invariant subset $X \subseteq \mathscr{V}(d,n)$ such that $X = \overline{X} \cap \mathscr{V}(d,n)$ where $\overline{X}$ is the Zariski closure of X [Reichstein & Vonessen, 2007, 3.1].

Theorem

[Reichstein & Vonessen, 2007, 8.1]Every irreducible generically free algebraic G-space is equivariantly birationally equivalent to an irreducible n-variety in $\mathscr{V}(d,n)$ for a suitable d.

Corollary

Every irreducible, algebraic principal G-bundle may be "realized" as a subbundle of $\mathfrak{P}(d,n)$ for suitable d.

Two Problems

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2. We want to study $\Gamma_h(Q_0(d,n),\mathfrak{M}_n)$ and $\Gamma_c(Q_0(d,n),\mathfrak{M}_n)$, and we would like $\Gamma_c(X,\mathfrak{M}_n)$ to be a C^*-algebra whenever $X \subseteq Q_0(d,n)$ is compact. However, there is no evident $*$-structure on $\mathfrak{M}_n$.

A Solution to Problems: Hypernormal points

Definition

A point $\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d)$ is called hypernormal in case $\sum_{i=1}^{d}[Z_i, Z_i^*]=0$. $\mathscr{H}\mathscr{N}(d,n) :=$ all hypernormal points.

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Kempf-Ness Theorem

$Q(d,n) \simeq \mathscr{H}\mathscr{N}(d,n)/K$, where $K := PU(n,\mathbb{C})$. [Le Bruyn, 2008, Theorem 2.11].

- Study $Q(d,n)$ using [Pearcy, 1962b, Pearcy, 1962a] and [Radjavi, 1968].

Intermezzo: A Riff on James Pascoe's Talk

- Define $p : M_n^d \times G \to [0, \infty)$ by $p(\mathfrak{z}, s) := \|s^{-1}\mathfrak{z}s\|$.

Theorem

T.F.A.E. for $\mathfrak{z} \neq 0$.

1. *$\mathfrak{z}G$ is a closed orbit.*
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   Further, T.F.A.E.
4. *$p(\mathfrak{z}, \cdot)$ achieves its minimum at I.*
5. *$\mathfrak{z}$ is a hypernormal point.*
6. *$\mathfrak{z}$ generates the W^*-algebra $\{s \in K \mid s^{-1}\mathfrak{z}s = \mathfrak{z}\}'$.*

Corollary

Theorem

$\mathscr{H}\mathscr{N}(d,n) \cap \mathscr{V}(d,n)$ is the bundle space of a principal $K := PU(n, \mathbb{C})$ bundle $\mathfrak{Q}$ over $Q_0(d,n)$. The bundle map is the restriction of π.

- Thus, $\mathfrak{Q}$ is a reduction [Steenrod, 1951, 9.4] (also restriction [Husemoller, 1994, 6.2]) of $\mathfrak{P}$ to K.

Corollary

The bundles $\mathfrak{M}_n$ and $\mathfrak{Q} \times_K M_n$ are isomorphic bundles over $Q_0(d,n)$ as topological bundles, and $\mathfrak{Q} \times_K M_n$ carries a natural $*$-structure.

Remarks

- There are other reductions of $\mathfrak{P}$ to K-bundles and any two give isomorphic $*$-structures on $\mathfrak{M}_n$, i.e., if $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are two reductions and if $\mathfrak{M}_n^{(i)} := \mathfrak{Q}_i \times_K M_n$, then $\mathfrak{M}_n^{(1)}$ is $*$-isomorphic to $\mathfrak{M}_n^{(2)}$. [Tomiyama & Takesaki, 1961, Thms 7 & 8]

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- Consequently, if $X \subset Q_0(d,n)$ is compact, then the C^*-algebras $\Gamma_c(X, \mathfrak{M}_n^{(1)})$ and $\Gamma_c(X, \mathfrak{M}_n^{(2)})$ are $*$-isomorphic. However, there may be no C^*-isomorphism that carries the image of $\Gamma_h(X, \mathfrak{M}_n)$ in one to the image of $\Gamma_h(X, \mathfrak{M}_n)$ in the other.

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- Consequently, if $X \subset Q_0(d, n)$ is compact, then the C^*-algebras $\Gamma_c(X, \mathfrak{M}_n^{(1)})$ and $\Gamma_c(X, \mathfrak{M}_n^{(2)})$ are $*$-isomorphic. However, there may be no C^*-isomorphism that carries the image of $\Gamma_h(X, \mathfrak{M}_n)$ in one to the image of $\Gamma_h(X, \mathfrak{M}_n)$ in the other.
- Is there a preferred reduction?

1. $\mathfrak{Q}$ - a reduction of $\mathfrak{P}$ to K, $\mathfrak{M}_n^* := \mathfrak{Q}[M_n]$, the M_n-bundle over $Q_0(d,n)$ associated to $\mathfrak{Q}$.
2. $\mathscr{D}$ - a domain such that $\overline{\mathscr{D}}$ is a Stein compact subset of $Q_0(d,n)$.
3. $A(\mathscr{D})$ - the sup-norm closure of $Hol(\overline{\mathscr{D}})$ in $C(\overline{\mathscr{D}})$.
4. $\partial\mathscr{D}$ - the Shilov boundary of $\mathscr{D}$, $\partial_e\mathscr{D}$ - the Choquet (or extreme) boundary of $\mathscr{D}$.
5. $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$ - the closure $\Gamma_h(\overline{\mathscr{D}}, \mathfrak{M}_n)$ in the C^*-algebra $\Gamma_c(\partial\mathscr{D}, \mathfrak{M}_n^*)$.

The Boundary Theorem

Theorem (Griesenauer, M and Solel)

Every point in $\partial_e \mathscr{D}$ gives rise, through evaluation, to a boundary representation of $\Gamma_c(\partial\mathscr{D}, \mathfrak{M}_n^*)$ for $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$, and so $\Gamma_c(\partial\mathscr{D}, \mathfrak{M}_n^*)$ is the C^*-envelope of $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$.

Fundamental Properties of $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$

Theorem (Griesenauer, M and Solel)

Suppose $\mathscr{D}$ is a domain in $Q_0(d, n)$ such that $\overline{\mathscr{D}}$ is Stein compact, then

1. The center of $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$ is $A(\mathscr{D})$ - independently of the reduction $\mathfrak{Q}$.
2. $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$ is a rank n Azumaya algebra.

Azumaya Algebras

Definition

Let A be a unital algebra with center C.

1. A is an Azumaya algebra over C iff
    1. A_C is a finitely generated, projective C-module.
    2. $\theta : A \otimes A^{\mathrm{op}} \to End(A_C)$, defined by $\theta(a \otimes b)(x) := axb$, is an isomorphism.

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2. A is an $\mathscr{A}_n$ -algebra iff
   1. A satisfies the identities of the $n \times n$ matrices.
   2. There are no (nonzero) homomorphisms of A in $M_{n-1}(\mathbb{C})$.

- $\mathscr{A}$ is for M. Artin.

The Formanek Center

Definition

The Formanek center of $\mathbb{G}_0(d,n)$:
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Example

$p(\mathfrak{z}) := (Z_1Z_2 - Z_2Z_1)^2 = -\det(Z_1Z_2 - Z_2Z_1)I$, $\mathfrak{z} \in M_2^2$, by Cayley-Hamilton, so $p \in \mathbb{F}(2,2)$.

Definition

If A satisfies the identities of the $n \times n$ matrices, then the Formanek center of A is
$F(A) := \cup_{d \geq 1}\{p(a_1, a_2, \cdots, a_d) \mid a_i \in A, p \in \mathbb{F}(d,n)\}$.

- $F(A) \subseteq \mathfrak{Z}A$.

Artin-Procesi Theorem and a lemma of Reichstein and Vonessen

Theorem

(See [Procesi, 1973, VII.2.1]) Suppose A is a unital algebra satisfying the identities of the $n \times n$ matrices. Then the following assertions are equivalent:

1. *A is an Azumaya algebra.*
2. *A is an $\mathscr{A}_n$ algebra.*
3. *$1_A \in F(A)A$.*

Artin-Procesi Theorem and a lemma of Reichstein and Vonessen

Theorem

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1. *A is an Azumaya algebra.*
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3. $1_A \in F(A)A$.

Lemma

[Reichstein & Vonessen, 2007, 2.10] Given $\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_r \in \mathscr{V}(d,n)$ there is a $p \in \mathbb{F}(d,n)$ such that $p(\mathfrak{z}_i) \neq 0$ for all i.

Proof that $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$ is Azumaya

- Recall: The center of $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$ is $A(\mathscr{D}) :=$ the sup-norm closure of $Hol(\overline{\mathscr{D}})$.
- For each $x \in \overline{\mathscr{D}}$, choose $\mathfrak{z} \in \mathscr{V}(d,n)$ such that $\pi(\mathfrak{z}) = x$ and choose $p \in \mathbb{F}(d,n)$ such that $p(\mathfrak{z}) \neq 0$.
- Since p is invariant, there is a regular function $\widetilde{p}$ on $Q(d,n)$, such that $\widetilde{p}(x) = p(\mathfrak{z}) \neq 0$.
- Since $\overline{\mathscr{D}}$ is compact, there is a finite family $\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_r \in \widetilde{\mathbb{F}(d,n)} \subseteq \mathfrak{Z}A(\partial\mathscr{D}, \mathfrak{M}_n^*) = A(\mathscr{D})$, which have no common zero on $\overline{\mathscr{D}}$.
- Therefore $1 \in \widetilde{p}_1 A(\mathscr{D}) + \widetilde{p}_2 A(\mathscr{D}) + \cdots + \widetilde{p}_r A(\mathscr{D})$. So by the Artin-Procesi theorem, $A(\partial\mathscr{D}, \mathfrak{M}_n^*)$ is Azumaya.

A final word - for now

- Many of the results described here have natural generalizations covering some of the matricial functions that Baruch and I introduced in [Muhly & Solel, 2013], particularly in the context of quiver algebras.
- The proper places where our matricial concomitants live is on what are known as quiver varieties.
- These, in turn, have simplectic reductions that are much like $\mathscr{H}\mathscr{N}(d,n)/K$.
- There are new challenges, however, and how they will unfold remains to be seen.

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