Operator algebras associated with monomial ideals

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Operator algebras of C*-correspondences I

Let E - a C*-correspondence over a C*-algebra A, $\varphi = \varphi_E : A \to \mathcal{L}(E)$.
A **representation** (π, t) of E on H is pair

- $\pi : A \to B(H)$
- $t : E \to B(H)$ s.t. $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle)$, $t(\varphi(a)\xi) = \pi(a)t(\xi)$.

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The **Toeplitz-Pimsner** algebra $\mathcal{T}_E$ is the C*-algebra generated by a universal representation, i.e., the universal C*-algebra containing E and A.

The **Tensor algebra** $\mathcal{T}_E^+$ is the non-selfadjoint subalgebra of $\mathcal{T}_E$ generated by A and E.
(There is a concrete version due to Pimsner and Muhly-Solel, we'll see below).

Operator algebras of C*-correspondences II

With every (π, t) there comes another representation $\psi_t : \mathcal{K}(E) \to B(H)$ given by

$$\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*.$$

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Let $J \subseteq \varphi^{-1}(\mathcal{K}(E))$. A representation (π, t) is said to be J-**covariant** if

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The J-**relative Cuntz-Pimnser algebra** $\mathcal{O}(J, E)$ is the universal C*-algebra generated by J-covariant representations.

The **Cuntz-Pimsner algebra** is $\mathcal{O}_E := \mathcal{O}(J_E, E)$, where

$$J_E = \ker \varphi_E^{\perp} \cap \varphi_E^{-1}(\mathcal{K}(E)).$$

(Here too there is a concrete version, we'll see below)

The operator algebras coming from C*-correspondences allow a unified treatment of a very broad spectrum of C*-algebras (graphs, dynamical systems, Cuntz-Krieger,...) and have a rich theory (GIUT, conditions for nuclearity, C*-envelopes,...)

Subproduct systems

A **subproduct system** is a family $X = \{X(n)\}_{n \in \mathbb{N}}$ of C*-correspondences (over a C*-algebra A) such that $X(0) = A$ and

$$X(m+n) \subseteq X(m) \otimes X(n)$$

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We construct the **Fock space**

$$\mathcal{F}(E) = A \oplus X(1) \oplus X(2) \oplus \dots$$

and the "representation" (φ_∞, T)

$$\varphi_\infty(a) = \varphi(a) \oplus (\varphi(a) \otimes I) \oplus (\varphi(a) \otimes I \otimes I) \oplus \dots$$

and , for $\xi \in X(1)$, $\eta \in X(n)$

$$T(\xi)\eta = P_{E^{\otimes n+1} \to X(n+1)} \big[\xi \otimes \eta\big].$$

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The **Cuntz-Pimsner algebra** of X is

$$\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{I},$$

where $\mathcal{I}$ is a certain ideal ($\mathcal{I} = \mathcal{K}(\mathcal{F}(X))$ in nice cases).

Operator algebras from subproduct systems II

If E is a C*-correspondence, then $X = \{E^{\otimes n}\}_{n \in \mathbb{N}}$ is a subproduct system. Then $\mathcal{T}(X) \cong \mathcal{T}_E$, $\mathcal{T}^+(X) \cong \mathcal{T}_E^+$ and $\mathcal{O}(X) \cong \mathcal{O}_E$, so all of the previously considered algebras fit this framewrok.

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On the other hand, for the C*-algebras there is no universal construction, likewise there is no Gauge Invariant Uniqueness Theorem. Some tools disappear from our tool-box, and techinal difficulties arise.

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For a subproduct system $X = \{E^{\otimes n}\}_{n \in \mathbb{N}}$ (that is, a product system) the answer to the first three is known, and a lot is known about the fourth too. For general subproduct systems, much less is known, even when $A = \mathbb{C}$.

A particular class of subproduct systems

Let $\mathbb{C}\langle z \rangle = \mathbb{C}\langle z_1, \ldots, z_d \rangle$ denote the algebra of polynomials over $\mathbb{C}$ in d noncommuting variables, and $\mathcal{I} = \mathcal{I}^{(1)} \oplus \mathcal{I}^{(2)} \oplus \ldots$ a homogeneous ideal in $\mathbb{C}\langle z \rangle$.

Put $E = \mathbb{C}^d$, and idenitify $\mathbb{C}\langle z \rangle \subseteq \mathcal{F}(E) = \mathbb{C} \oplus E \oplus E^{\otimes n} \oplus \ldots$.

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X is a subproduct system that encodes very well the polynomial relations in the ideal $\mathcal{I}$:

Theorem (Popoescu, S.-Solel)

$\mathcal{T}^+(X)$ *is the universal unital operator algebra generated by a row contraction satisfying the relations in* $\mathcal{I}$.

The setting

We will henceforth restrict attention to the case where $\mathcal{I}$ is generated by **monomials**.
Fix a basis $\{e_1, \dots, e_d\}$, and denote $T_i = T(e_i)$.
To prevent confusion we denote

$$C^*(T) = \mathcal{T}(X) = C^*(I, T_1, \dots, T_d)$$

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Thus T is a row contraction and $p(T) = 0$ for all $p \in \mathcal{I}$. Every such row contraction determines a UCC representation of $\mathcal{A}_X$. Finally:

$$C^*(T)/\mathcal{K} = \mathcal{O}(X).$$

Some formulas

For $\mu = \mu_1 \cdots \mu_k \in \mathbb{F}_d^+$ we write

$$z^\mu = z_{\mu_1} \cdots z_{\mu_k} \ , \ e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}.$$

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$T_\mu^* T_\mu$ commutes with $T_\nu^* T_\nu$ for all μ, ν.

Example: Subshifts

On $\{1, \ldots, d\}^{\mathbb{Z}}$, let σ denote the left shift.
A shift invariant space $\Lambda \subseteq \{1, \ldots, d\}^{\mathbb{Z}}$ is called a subshift.
A subshift Λ is determined by the set of $\mathfrak{F}$ of forbidden words.

$$\mathfrak{F} = \text{ forbidden words } , \Lambda^* = \text{ allowed words}$$

From a subshift we can construct an ideal $\mathcal{I}_\Lambda$ generated by z^μ, for $\mu \in \mathfrak{F}$.
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In this setting, the algebras $C^*(T)/\mathcal{K}$ were studied by Matsumoto, and were called **subshift C*-algebras**.
However, Matsumoto later changed his definitions (in order that his theorems remain true).

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The situation we are dealing with has nature particularly tractable to computations (e.g., $T_1, \dots, T_d$ are mutually orthogonal partial isometries), and we are able to combine the two theories.

What is the smallest C*-correspondence containing T?

If we form a C*-correspondence E over A containing $T_1, \ldots, T_d$, then

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Thus $T_j T_i^* T_i \in E$, and $T_i^* T_i T_j$.

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(where $T_{ij} = T_i T_j$) thus

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So $T_{ij}^* T_{ij} \in A$.
Likewise, $T_\mu^* T_\mu \in A$ for all μ.

The C*-correspondence of a monomial ideal

We define

$$A = C^*(I, T_\mu^* T_\mu : \mu \in \mathbb{F}_+^d).$$

and

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Note that A is a commutative C*-algebra.

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Note that A is a commutative C*-algebra.

We now cosider the algebras $\mathcal{T}_E$, $\mathcal{T}_E^+$ and the (relative) Cuntz-Pimsner algebras $\mathcal{O}(J, E)$.

Orientating the algebras I

Let ${}_A E_A$ be the C*-correspondence of a monomial ideal $\mathcal{I}$ in $\mathbb{C}\langle z_1, \dots, z_d \rangle$.

Theorem

$C^*(T)$ is the relative Cuntz-Pimsner algebra $\mathcal{O}(J, E)$ for the ideal J generated by $\{1 - T_\mu^* T_\mu \mid \mu \in \mathbb{F}_d^+\}$.

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In particular $J \subseteq J_E \subseteq A$, and there are canonical $*$-epimorphisms

$$\mathcal{T}_E \to C^*(T) \to \mathcal{O}_E \to C^*(T)/\mathcal{K}(\mathcal{F}_X).$$

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Corollary (C*-correspondences help)

The algebras $C^*(T) = \mathcal{T}(X)$ and $C^*(T)/\mathcal{K}(\mathcal{F}_X) = \mathcal{O}(X)$ are nuclear.

Orientating the algebras II

Theorem

The following diagram holds:

$$C^*(T) \not\simeq \mathcal{T}_E \Leftrightarrow \mathcal{I} \neq (0) \Leftrightarrow E \not\simeq \mathbb{C}^d \qquad\qquad C^*(T) \simeq \mathcal{T}_E \Leftrightarrow \mathcal{I} = 0 \Leftrightarrow E \simeq \mathbb{C}^d$$

$$\ker \phi_E \neq 0 \qquad\qquad \ker \phi_E = 0 \qquad\qquad \Downarrow$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \mathcal{O}_E \simeq C^*(T)/\mathcal{K}(\mathcal{F}_X) \simeq \mathcal{O}_d \,,\ \ker \phi_E = 0$$

$$\mathcal{O}_E \simeq C^*(T) \qquad \mathcal{O}_E \simeq C^*(T)/\mathcal{K}(\mathcal{F}_X)$$

with the understanding that all $*$-isomorphisms are canonical.

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We aslo have precise combinatorial conditions for when $\ker \phi_E = 0$.

Proposition

Let ${}_A E_A$ be the C*-correspondence of a monomial ideal $\mathcal{I} \triangleleft \mathbb{C}\langle z_1, \dots, z_d \rangle$. The following are equivalent:

1. $P_\emptyset \in A$;
2. $\ker \phi_E = \mathbb{C} \cdot P_\emptyset$;
3. **$\ker \boldsymbol{\phi_E} \neq \mathbf{(0)}$**;
4. **for every $\boldsymbol{i = 1, \dots, d}$ there is a $\boldsymbol{\mu_i} \in \boldsymbol{\Lambda^*}$ such that $\boldsymbol{\mu_i i} \notin \boldsymbol{\Lambda^*}$**;
5. for every $i = 1, \dots, d$ there is a $\mu_i \in \mathbb{F}_+^d$ such that $z^{\mu_i} \notin \mathcal{I}$ and $z^{\mu_i} z_i \in \mathcal{I}$;
6. $J_E := \ker \phi_E^\perp \cap \phi^{-1}(\mathcal{K}(E)) = \langle 1 - P_\emptyset \rangle = A(1 - P_\emptyset)$;
7. $1 \notin J_E$.

If these conditions hold then $\ker \phi_E = \langle T_{\mu_1}^* T_{\mu_1} \cdots T_{\mu_d}^* T_{\mu_d} \rangle$ for any tuple of words $(\mu_1, \dots, \mu_d)$ such that $\mu_i i \notin \Lambda^*$ for all $i = 1, \dots, d$.

C*-envelopes I

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Let E be the C-correspondence of a monomial ideal $\mathcal{I} \lhd \mathbb{C}\langle z_1, \ldots, z_d \rangle$. Then the tensor algebra $\mathcal{T}_E^+$ is hyperrigid (in $\mathcal{O}_E$).*

C*-envelopes I

A theorem of Katsoulis-Kribs (following Muhly-Solel) says that

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Recall: $\mathcal{A} \subseteq \mathcal{B} = C^*(\mathcal{A})$ is said to be **hyperrigid** if for every unital $\pi : \mathcal{B} \to B(H)$, the UCC map $\pi\big|_{\mathcal{A}}$ has the **uniqe extension property**.

C*-envelopes II

We now turn to the algebra $\mathcal{A}_X$.
Here there is no general theory to help us and our results are far from final form.

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Theorem

Let X be the subproduct system of a monomial ideal $\mathcal{I} \triangleleft \mathbb{C}\langle z_1, \dots, z_d \rangle$ **of finite type** *and let $q \colon C^*(T) \to C^*(T)/\mathcal{K}(\mathcal{F}_X)$. Then $q(\mathcal{A}_X)$ is hyperrigid in $C^*(T)/\mathcal{K}(\mathcal{F}_X)$, hence $C_e^*(q(\mathcal{A}_X)) = C^*(T)/\mathcal{K}(\mathcal{F}_X)$.*

C*-envelopes III

Theorem

Let X be the subproduct system of a monomial ideal $\mathcal{I} \lhd \mathbb{C}\langle z_1, \dots, z_d \rangle$ **of finite type** *and let $q \colon C^*(T) \to C^*(T)/\mathcal{K}(\mathcal{F}_X)$. Then*

$q|_{\mathcal{A}_X}$ *is not completely isometric*

$\Updownarrow$ (1)

$$C^*_e(\mathcal{A}_X) \simeq C^*(T)$$

$\Downarrow$ (3) $\Uparrow$ (4)

$$\forall i = 1, \dots, d, \exists \mu_i \in \Lambda^*. \mu_i i \notin \Lambda^*$$

$q|_{\mathcal{A}_X}$ *is completely isometric*

$\Updownarrow$ (2)

$$C^*_e(\mathcal{A}_X) \simeq C^*(T)/\mathcal{K}(\mathcal{F}_X)$$

$\Downarrow$ (4) $\Uparrow$ (3)

$$\exists i \in \{1, \dots, d\}, \forall \mu \in \Lambda^*. \mu i \in \Lambda^*$$

Item (4) holds under the assumption that the μ_is can always be chosen to have the same length. In particular item (4) holds when $X = X_\Lambda$.

C*-correspondences help, again

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(2) follows from the previous theorem.

(3) (Going up). The conidition implies $\ker \phi = (0)$ by proposition, hence by

$$C^*(T)/\mathcal{K} = \mathcal{O}_E = C^*_e(\mathcal{T}_E^+)$$

where first equality follows from a previous theorem.
But $\mathcal{A}_X \hookrightarrow \mathcal{T}_E^+$. Thus $q\big|_{\mathcal{A}_X}$ is completely isometric, and (3) follows. □

Remark

In the previous theorem we say that (under assumption of finite type)

$$C_e^*(\mathcal{A}_X) = C^*(T) = \mathcal{T}(X) \text{ or } C_e^*(\mathcal{A}_X) = C^*(T)/\mathcal{K} = \mathcal{O}(X).$$

In all known examples until now, we had either

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Recently Dor-On and Markiewicz showed that these are **not** the only possibilities.

Universal property

Theorem

Let $\mathcal{I}$ be a monomial ideal of finite type k. Then the algebra $C^*(T)/\mathcal{K}(\mathcal{F}_X)$ is the universal C-algebra generated by a row contraction $s = [s_1, \ldots, s_d]$ such that*

1. $I = \sum_{i=1}^{d} s_i s_i^*$;
2. $p(s) = 0$ *for all* $p \in \mathcal{I}$;
3. $s_i^* s_i = \sum_{\mu \in E_i^k} s_\mu s_\mu^*$ *where* $E_i^k = \{\mu \in \Lambda_k^* \mid i\mu \in \Lambda^*\}$, *for all* $i = 1, \ldots, d$.

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Previously appeared in joint work with Solel, though proof there seems to have a gap.

Application of NSA methods

Proof.

Given such a tuple $S_1, \ldots, S_d \in B(H)$, need to construct

$$\pi : C^*(T)/\mathcal{K} \to B(H) \ , \ \pi : q(T_i) \mapsto S_i.$$

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By previous results (Popescu, S.-Solel) construct $\pi : q(\mathcal{A}_X) \to B(H)$,

$$\pi : q(T_i) \mapsto S_i \ , \ i = 1, \dots, d.$$

Then, using techniques of previous proposition, show that π has the **unique extension property**.
Thus π extends to $*$-representation, as required. □

Classification I

Theorem

Let X and Y be subproduct systems associated with the homogeneous ideals $\mathcal{I} \lhd \mathbb{C}\langle x_1, \dots, x_d \rangle$ and $\mathcal{J} \lhd \mathbb{C}\langle y_1, \dots, y_{d'} \rangle$. Without loss of generality suppose that $x_i \notin \mathcal{I}$ and $y_j \notin \mathcal{J}$ for all i, j. The following are equivalent:

1. *$\mathcal{A}_X$ and $\mathcal{A}_Y$ are completely isometrically isomorphic;*
2. *$\mathcal{A}_X$ and $\mathcal{A}_Y$ are algebraically isomorphic;*
3. *X and Y are similar;*
4. *X and Y are isomorphic;*

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Explanation: $X \simeq Y$ iff $U_m : X_n \to Y_n$ (unitaries) such that

$$U_{m+n}(P_{m+n}(x_m \otimes x_n)) = P_{m+n}(U_m(x_m) \otimes U_n(x_n))$$

The quantised dynamics I

On the commutative C*-algebra $A = C^*(T_\mu^* T_\mu \mid \mu \in \Lambda^*)$ we define d $*$-endomorphisms

$$\alpha_i : A \to A,$$

$$\alpha_i(a) = T_i^* a T_i.$$

We call the system $(A, \alpha_1, \dots, \alpha_d)$ the **quantised dynamical system** of the allowable words.

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Theorem

The quatised dyamical system's conjugacy class is a complete invariant of the monomial ideal.

Indeed, $\alpha_{\mu_1} \circ \cdots \circ \alpha_{\mu_k}(I) = 0$ determines the monomials z^μ in the ideal.

The quantised dynamics II

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We obtain a partially defined classical dynamical system $(\Omega, \varphi_1, \ldots, \varphi_d)$, where each φ_i is only defined on $\Omega_i \subseteq \Omega$.

Classification II

Theorem

Let $\mathcal{I} \lhd \mathbb{C}\langle x_1, \dots, x_d \rangle$ and $\mathcal{J} \lhd \mathbb{C}\langle y_1, \dots, y_{d'} \rangle$ be monomial ideals. Let E_A and F_B be the C-correspondences associated with $\mathcal{I}$ and $\mathcal{J}$, respectively. Furthermore let $(\Omega_{\mathcal{I}}, \varphi)$ and $(\Omega_{\mathcal{J}}, \psi))$ be the corresponding quantised dynamics. The following are equivalent:*

1. *$\mathcal{T}_E^+$ and $\mathcal{T}_F^+$ are completely isometrically isomorphic;*
2. *$\mathcal{T}_E^+$ and $\mathcal{T}_F^+$ are isomorphic as topological algebras;*
3. *$(\Omega_{\mathcal{I}}, \varphi)$ and $(\Omega_{\mathcal{J}}, \psi)$ are locally (piecewise) conjugate;*
4. *E and F are unitarily equivalent.*

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Locally (piecewise) conjugate: Mix between the notion of Davidson-Katsoulis for dynamical systems and the notion of Davidson-Roydor for topological graphs.

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Proof:

This follows from Davidson-Roydor, because a partial dynamical system is a topological graph. We also present an alternative proof.

Thank you!