Commuting Dilations and Linear Positivstellensätze

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Multivariate Operator Theory

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based on joint work with

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Outline

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- Linear Pencil
- Spectrahedra
- LMI Domination Problem

Relax: Free Analysis meets LMIs
- Free Spectrahedra
- Relaxed LMI Domination
- Linear Positivstellensätze

Commuting Dilations
- Free LMI Domination and Dilation Theory

Apply: Matrix Cube Problem
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- Coin Tossing

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Roadmap

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Linear Pencil, Linear Matrix Inequality (LMI)

- For symmetric matrices $A_0, A_1, \ldots, A_s \in \mathbb{S}^{d\times d}$ and $x = (x_1, \ldots, x_s)$, the expression

$$L(x) = A_0 + A_1 x_1 + \cdots + A_s x_s$$

is called a $d \times d$ linear pencil.

If $A_0 = I$, we say that $L(x)$ is monic.

- A linear matrix inequality (LMI) is of the form $L(x) \succeq 0$.

Its scalar solution set

$$\begin{aligned}\mathcal{D}_L(1) &= \{x \in \mathbb{R}^s \mid L(x) \succeq 0\} \\ &= \{x \in \mathbb{R}^s \mid A_0 + A_1 x_1 + \cdots + A_s x_s \succeq 0\}\end{aligned}$$

is called an LMI domain or also a spectrahedron.

It is a convex and semialgebraic subset of $\mathbb{R}^s$.

LMIs: Their importance

Applications

- control theory: many optimization problems in control theory, system identification and signal processing can be formulated using LMIs
- convex optimization: optimizing a linear objective function over a spectrahedron is a semidefinite program and can be solved efficiently using interior point methods
- real algebraic geometry: positive polynomials, convexity
- combinatorics and graph theory: Goemans-Williamson MAX CUT approximation algorithm
- operator theory: complete positivity, matrix convex sets, ...

LMI Domination

Inclusion of spectrahedra

Given linear pencils $L(x)$ and $\tilde{L}(x)$ in the same number of variables:

When does one dominate the other?

That is, does $L(x) \succeq 0$ *imply* $\tilde{L}(x) \succeq 0$*?*

$$\mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1) \text{ ?}$$

Roadmap

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Free Analysis enters

Relax. Be Free

Given a $d \times d$ linear pencil

$$L(x) = A_0 + A_1 x_1 + \cdots + A_s x_s,$$

it is natural to substitute symmetric matrices X_j for the variables x_j:

- For $X = (X_1, \ldots, X_s) \in (\mathbb{S}^{n \times n})^s$, the evaluation $L(X)$ is

$$L(X) := A_0 \otimes I_n + A_1 \otimes X_1 + \cdots + A_s \otimes X_s \in \mathbb{S}^{dn \times dn}.$$

The tensor product in this expression is the standard (Kronecker) tensor product of matrices.

Free spectrahedron

- For each dimension $n \in \mathbb{N}$,

$$\mathcal{D}_L(n) := \left\{X \in (\mathbb{S}^{n\times n})^s \mid L(X) \succeq 0\right\}$$

  are natural tightenings of $\mathcal{D}_L(1)$.

- The supreme tightening: free (or quantum) spectrahedron is

$$\mathcal{D}_L := \bigcup_{n\in\mathbb{N}} \mathcal{D}_L(n).$$

Relaxed LMI Domination

Inclusion of free spectrahedra

Given linear pencils $L(x)$ and $\tilde{L}(x)$ in the same number of variables:

When does one freely dominate the other?

That is, does $L(X) \succeq 0$ *imply* $\tilde{L}(X) \succeq 0$*?*

$$\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \text{ ?}$$

Relaxed LMI Domination

Inclusion of free spectrahedra

Given linear pencils $L(x)$ and $\tilde{L}(x)$ in the same number of variables:

When does one freely dominate the other?

That is, does $L(X) \succeq 0$ *imply* $\tilde{L}(X) \succeq 0$*?*

$$\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \text{ ?}$$

Note: $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \;\Rightarrow\; \mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1)$.

In general, $\mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1) \;\not\Rightarrow\; \mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$.

Relaxed LMI Domination

Given two linear pencils

$$L(x) = I + \sum_{j=1}^{s} x_j A_j, \qquad \tilde{L}(x) = I + \sum_{j=1}^{s} x_j \tilde{A}_j.$$

Theorem (Linear Positivstellensatz)

Suppose $\mathcal{D}_L(1)$ is bounded. Then the following are equivalent:

(i) $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$;

(ii) $\tilde{L}(x) = \sum_{j=1}^{\mu} V_j^T L(x) V_j$, *where* $\sum_{j=1}^{\mu} V_j^T V_j = I$.

Relaxed LMI Domination

Given two linear pencils

$$L(x) = I + \sum_{j=1}^{s} x_j A_j, \qquad \tilde{L}(x) = I + \sum_{j=1}^{s} x_j \tilde{A}_j,$$

consider the unital linear map

$$\tau : \mathcal{S} \longrightarrow \tilde{\mathcal{S}}, \qquad A_j \longmapsto \tilde{A}_j.$$

Here $\mathcal{S} = \operatorname{span}\{I, A_1, \dots, A_s\}$ and similarly for $\tilde{\mathcal{S}}$.

Theorem (Linear Positivstellensatz)

Suppose $\mathcal{D}_L(1)$ is bounded. Then the following are equivalent:

(i) $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$;

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(iii) τ *is completely positive.*

Relaxed LMI Domination

An Algorithm

Given $L, \tilde{L}$, how does one check whether $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$ holds?

This is equivalent to an explicit LMI:

- Define the linear map $\tau : \operatorname{ran} L \longrightarrow \operatorname{ran} \tilde{L}$, i.e.,

$$\tau(L(x)) = \tilde{L}(x) \quad \text{for all } x \in \mathbb{R}^s.$$

- Then $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$ if and only if τ extends to a linear map $\hat{\tau} : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}^{\tilde{d}\times\tilde{d}}$ making the Choi matrix

$$C = \begin{bmatrix} \hat{\tau}(E_{11}) & \cdots & \hat{\tau}(E_{1d}) \\ \vdots & \ddots & \vdots \\ \hat{\tau}(E_{d1}) & \cdots & \hat{\tau}(E_{dd}) \end{bmatrix} \in \mathbb{R}^{d\tilde{d}\times d\tilde{d}}$$

positive semidefinite. Here, E_{ij} denote the $d \times d$ matrix units.

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The error . . .

. . . of the free relaxation

- $\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$ is a semidefinite program and implies $\mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1)$.
- The worst case error? What is the smallest $\rho = \rho(L, \tilde{d})$, depending only upon the size $\tilde{d}$ of $\tilde{L}$, such that

$$\rho\, \mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1) \implies \mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}?$$

We call ρ the $\mathcal{D}_L$-inclusion scale.

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☞ Dilation theory provides the answer.

ℹ Recall: C dilates to T if

$$T = \begin{pmatrix} C & * \\ * & * \end{pmatrix}.$$

Then we say T is a compression of C.

Dilation theory . . .

. . . and the error of the free relaxation

- The largest $\gamma = \gamma(L, n)$ such that for each $X \in \mathcal{D}_L(n)$ the tuple γX dilates to a commuting tuple of self-adjoint operators with joint spectrum in $\mathcal{D}_L(1)$ is called the $\mathcal{D}_L$-commutability index.

Dilation theory ...

... and the error of the free relaxation

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¿Puzzle?

Let $n \in \mathbb{N}$.

- Can you dilate all $n \times n$ symmetric matrices to commuting self-adjoint operators?

Dilation theory . . .

. . . and the error of the free relaxation

- The largest $\gamma = \gamma(L, n)$ such that for each $X \in \mathcal{D}_L(n)$ the tuple γX dilates to a commuting tuple of self-adjoint operators with joint spectrum in $\mathcal{D}_L(1)$ is called the $\mathcal{D}_L$-commutability index.

¿Puzzle?

Let $n \in \mathbb{N}$.

- Can you dilate all $n \times n$ symmetric matrices to commuting self-adjoint operators?
- Can you do this in a norm non-increasing way?

Dilation theory . . .

. . . and the error of the free relaxation

Theorem (Commutability index = Inclusion scale)

Suppose L is a monic linear pencil and $\mathcal{D}_L(1)$ is bounded.

(1) *The commutability index for L equals its inclusion scale, $\rho(L) = \gamma(L)$. That is $\gamma(L, \tilde{d})$ is the largest constant such that*

$$\gamma(L, \tilde{d})\, \mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}}$$

for each monic linear pencil $\tilde{L}$ of size $\tilde{d}$ satisfying $\mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1)$.

Dilation theory . . .

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Theorem (Commutability index = Inclusion scale)

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(1) *The commutability index for L equals its inclusion scale, $\rho(L) = \gamma(L)$. That is $\gamma(L, \tilde{d})$ is the largest constant such that*

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for each monic linear pencil $\tilde{L}$ of size $\tilde{d}$ satisfying $\mathcal{D}_L(1) \subseteq \mathcal{D}_{\tilde{L}}(1)$.

(2) *If $\mathcal{D}_L$ is symmetric about the origin (i.e., $-\mathcal{D}_L \subseteq \mathcal{D}_L$), then*

$$\gamma(L, d) \geq \frac{1}{d}.$$

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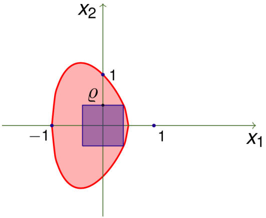
Matrix Cube Problem

Nemirovskii, ICM 2006

- Given a monic linear pencil L, does

$$\mathcal{D}_{\mathcal{C}}(1) = [-1,1]^s \subseteq \mathcal{D}_L(1)\ ?$$



- For the free cube $\mathcal{D}_{\mathcal{C}} = \{(X_1,\dots,X_s) : \|X_j\| \leq 1\}$ determining $\rho(\mathcal{C}, d)$ for the relaxed inclusion $\mathcal{D}_{\mathcal{C}} \subseteq \mathcal{D}_L$ is the matrix cube problem of Ben-Tal and Nemirovskii.
  - problems of robust control, e.g. Lyapunov stability analysis for uncertain dynamical systems;
  - various combinatorial problems can be reduced to maximizing a positive definite quadratic form over the unit cube;
  - the case $d = 2$ (really rank $A_j \leq 2$) contains the Nesterov $\frac{\pi}{2}$-Theorem about optimizing a positive definite quadratic form over the unit cube. Also the symmetric (little) Grothendieck inequality.

How good is the relaxation?

For $d \in \mathbb{N}$, define $\vartheta(d) \geq 1$ by

$$\frac{1}{\vartheta(d)} = \min_{\substack{a \in \mathbb{R}^d \\ |a_1|+\cdots+|a_d|=d}} \int_{S^{d-1}} \left| \sum_{i=1}^{d} a_i \xi_i^2 \right| d\xi = \min_{\substack{B \in \mathbb{S}^{d\times d} \\ \mathrm{trace}|B|=d}} \int_{S^{d-1}} |\xi^* B \xi| \, d\xi$$

Theorem (Ben-Tal & Nemirovskii; Helton, K, McCullough, Schweighofer)

If L is a $d \times d$ monic linear pencil with $[-\varrho, \varrho]^s = \mathcal{D}_{\mathcal{C}_\varrho}(1) \subseteq \mathcal{D}_L(1)$, then

$$\mathcal{D}_{\mathcal{C}_\varrho} \subseteq \vartheta(d)\, \mathcal{D}_L.$$

Moreover, $\vartheta(d)$ is the smallest number with this property:
if $\vartheta' < \vartheta(d)$, then there is a $d \times d$ pencil L such that $\mathcal{D}_{\mathcal{C}_\varrho}(1) \subseteq \mathcal{D}_L(1)$ but

$$\mathcal{D}_{\mathcal{C}_\varrho} \not\subseteq \vartheta'\, \mathcal{D}_L.$$

How good is the relaxation?

For $d \in 2\mathbb{N}$, define $\vartheta(d) \geq 1$ by

$$\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{d-1}}{d} \leq \frac{1}{\vartheta(d)} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{d}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{d}{4}+1\right)} \leq \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{d+1}}.$$

Recall: if $n \in \mathbb{N}$ then $\Gamma(n) = (n-1)!$ and $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$.

Theorem (Ben-Tal & Nemirovskii; Helton, K, McCullough, Schweighofer)

If L is a $d \times d$ monic linear pencil with $[-\varrho, \varrho]^s = \mathcal{D}_{\mathcal{C}_\varrho}(1) \subseteq \mathcal{D}_L(1)$, then

$$\mathcal{D}_{\mathcal{C}_\varrho} \subseteq \vartheta(d)\, \mathcal{D}_L.$$

Moreover, $\vartheta(d)$ is the smallest number with this property:
if $\vartheta' < \vartheta(d)$, then there is a $d \times d$ pencil L such that $\mathcal{D}_{\mathcal{C}_\varrho}(1) \subseteq \mathcal{D}_L(1)$ but

$$\mathcal{D}_{\mathcal{C}_\varrho} \not\subseteq \vartheta'\, \mathcal{D}_L.$$

How good is the relaxation?

Proof ingredients: dilation theory and probability

(1) The bound $\vartheta(d)$ is a corollary of a Dilation Theoretic result.

(2) Worst case analysis (inspired by dilation theory) shows the bound is sharp;

(3) New results for the beta distribution identify $\vartheta(d)$ analytically.

How good is the relaxation?

Proof ingredients. (1) Dilation theory

$$\frac{1}{\vartheta(d)} = \min_{\substack{B \in \mathbb{S}^{d\times d} \\ \mathrm{trace}|B| = d}} \int_{S^{d-1}} |\xi^* B \xi| \; d\xi \;\approx\; \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{d}}.$$

Theorem (Simultaneous Dilation)

Let $d \in \mathbb{N}$. There is a Hilbert space $\mathcal{H}$, a family $\mathscr{C}_d$ of commuting self-adjoint contractions on $\mathcal{H}$, and an isometry $V : \mathbb{R}^d \to \mathcal{H}$ such that for each symmetric $d \times d$ contraction matrix X there exists a $T \in \mathscr{C}_d$ such that

$$X = \vartheta(d)\; V^* T V.$$

Moreover, $\vartheta(d)$ is the smallest such constant.

How good is the relaxation?

Simultaneous dilation in action: Proof of the scaling theorem

Let the pencil L be of size d with $\mathcal{D}_{\mathcal{C}}(1) = [-1,1]^s \subseteq \mathcal{D}_L(1)$.

The claim is $\mathcal{D}_{\mathcal{C}} \subseteq \vartheta(d)\mathcal{D}_L$.

It suffices to prove $\mathcal{D}_{\mathcal{C}}(d) \subseteq \vartheta(d)\mathcal{D}_L(d)$. (cp)

For $X = (X_1, \dots, X_s) \in \mathcal{D}_{\mathcal{C}}(d)$, by the Simultaneous Dilation,

$$\frac{1}{\vartheta(d)} X = V^* T V = (V^* T_1 V, \dots, V^* T_s V), \qquad T_j \in \mathscr{C}_d.$$

Because the T_j commute and are contractions, $\mathcal{D}_{\mathcal{C}}(1) \subseteq \mathcal{D}_L(1)$ implies

$$L(T) \succeq 0. \qquad \text{(spectral theorem aka simultaneous diagonalization)}$$

Hence,

$$L\Big(\frac{1}{\vartheta(d)} X\Big) = (I \otimes V)^* \, L(T) \, (I \otimes V) \succeq 0.$$

□

How good is the relaxation?

Proof ingredients. (1) Dilation theory

Simultaneous Dilation Theorem. Let $d \in \mathbb{N}$. There is a Hilbert space $\mathcal{K}$, a family $\mathscr{C}_d$ of commuting self-adjoint contractions on $\mathcal{K}$, and an isometry $V : \mathbb{R}^d \to \mathcal{K}$ such that for each symmetric $d \times d$ contraction matrix X there exists a $T \in \mathscr{C}_d$ such that $X = \vartheta(d)\, V^* TV$. Moreover, $\vartheta(d) \approx \frac{\sqrt{d\pi}}{2}$ is the smallest such constant.

- $\mathcal{O}(d) \subseteq M_d(\mathbb{R})$ is the orthogonal group with its Haar measure dU;
- Let $\mathcal{K} = \mathbb{R}^d \otimes L^2(dU) = L^2_{\mathbb{R}^d}(dU)$ square integrable functions $f : \mathcal{O}(d) \to \mathbb{R}^d$;
- Define $V : \mathbb{R}^d \to \mathcal{K}$ by $Vx(U) = x$ (Vx = constant function x); $V^* f = \int_{\mathcal{O}(d)} f(U)\, dU$;
- $\mathscr{D}_d$ is the set of contractive $d \times d$ diagonal matrices;
- For $D : \mathcal{O}(d) \to \mathscr{D}_d$, define the twisted multiplication operator $M_D : \mathcal{K} \to \mathcal{K}$,
$$M_D f(U) = UD(U)U^*\, f(U);$$
- $\|M_D\| \leq \|D\|_\infty \leq 1$;
- If $E : \mathcal{O}(d) \to \mathscr{D}_d$, then $M_D M_E = M_E M_D$ (D and E pointwise commute);
- $\mathscr{C}_d = \{M_D \mid D : \mathcal{O}(d) \to \mathscr{D}_d\}$ is a collection of commuting self-adjoint contractions on $\mathcal{K}$.

How good is the relaxation?

Proof ingredients. (2) Worst case analysis

$$\frac{1}{\vartheta(d)} = \min_{\substack{a\in\mathbb{R}^d \\ |a_1|+\cdots+|a_d|=d}} \int_{S^{d-1}} \left|\sum_{i=1}^{d} a_i \xi_i^2\right| d\xi = \min_{\substack{B\in\mathbb{S}^{d\times d} \\ \mathrm{trace}|B|=d}} \int_{S^{d-1}} |\xi^* B \xi|\ d\xi.$$

Proposition (smoothing)

$$\frac{1}{\vartheta(d)} = \min_{\substack{s,t\in\mathbb{N},\ a,b\in\mathbb{R}_{\geq 0} \\ as+bt=d \\ s+t=d}} \int_{S^{d-1}} \left|\xi^* J(s,t;a,b)\xi\right| d\xi$$

$$= \min_{s,t,a,b} \frac{2}{d}\left(as\ I_{\frac{a}{a+b}}\left(\frac{t}{2}, \frac{s}{2}+1\right) + bt\ I_{\frac{b}{a+b}}\left(\frac{s}{2}, \frac{t}{2}+1\right)\right) - 1.$$

Here $J(s,t;a,b) = a\ I_s \oplus -b\ I_t$ is the diagonal matrix with first s diagonal entries a and last t diagonal entries $-b$. The regularized incomplete Beta function is

$$I_p(\alpha,\beta) = \frac{B_p(\alpha,\beta)}{B(\alpha,\beta)} = \frac{\int_0^p x^{\alpha-1}(1-x)^{\beta-1}dx}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx}.$$

How good is the relaxation?

Proof ingredients. (2) Worst case analysis: Equipoints

$$\frac{1}{\vartheta(d)} = \min_{s,t} \min_{a,b} \frac{2}{d} \left(as\, I_{\frac{a}{a+b}} \left(\frac{t}{2}, \frac{s}{2} + 1 \right) + bt\, I_{\frac{b}{a+b}} \left(\frac{s}{2}, \frac{t}{2} + 1 \right) \right) - 1.$$

$$\frac{1}{\vartheta(d)} = \min_{s+t=d} \left\{ 2 I_{e_{s,t}} \left(\frac{s}{2}, \frac{t}{2} + 1 \right) - 1 \right\}.$$

- For a given s, t, the equipoint $e_{s,t}$ is defined by

$$I_{e_{s,t}} \left(\frac{t}{2}, \frac{s}{2} + 1 \right) = 1 - I_{e_{s,t}} \left(\frac{s}{2}, \frac{t}{2} + 1 \right) = I_{1-e_{s,t}} \left(\frac{s}{2}, \frac{t}{2} + 1 \right).$$

- $e_{s,t} = \frac{a}{a+b}$ determines a, b (along with $as + bt = s + t$).
- Evidently $e_{s,s} = \frac{1}{2}$.

How good is the relaxation?

Proof ingredients. (3) Coin tossing

Perform $d \in \mathbb{N}$ independent flips of a biased coin whose probability of coming up heads is p. Let $\mathfrak{S}$ denote the random variable representing the number of heads which occur. We call $e_{s,d-s} \in [0,1]$ an equipoint of $s \in \mathbb{N}$, provided $P_{e_{s,d-s}}(\mathfrak{S} \geq s) = P_{e_{s,d-s}}(\mathfrak{S} \leq s)$.

Theorem (Simmons)

Let $s \in \mathbb{N}$. If $s \geq \frac{d}{2}$, then $\frac{s+1}{d+2} \leq e_{s,d-s} \leq \frac{s}{d}$.

Interpretation of Simmons' Theorem in plain English[1]

Toss a coin whose probability for head is $\frac{s}{d} \geq \frac{1}{2}$, d times. Then:

the probability of getting $< s$ heads

is smaller than

the probability of getting $> s$ heads.

[1] Simmons was looking at blocks of random digits and noticed that a given digit was more likely to appear fewer than the expected number of times than it was to appear more.

How good is the relaxation?

Wrapping up: Identifying $\vartheta(d)$

Todo: simplify

$$\frac{1}{\vartheta(d)} = \min_{s+t=d} \left\{ 2 I_{e_{s,t}}\left(\frac{s}{2}, \frac{t}{2}+1\right) - 1 \right\}.$$

For simplicity assume $d \in 2\mathbb{N}$. Without loss of generality, $s \geq t$.

Claim: minimum is attained when $s = t = \frac{d}{2}$.

Simmons' theorem (for half-integers) implies

$$f(s,t) := I_{e_{s,t}}\left(\frac{s}{2}, \frac{t}{2}+1\right)$$

is two-step monotone. This together with

$$f\left(\frac{d}{2}, \frac{d}{2}\right) \leq f\left(\frac{d}{2}+1, \frac{d}{2}-1\right)$$

yields the claim and suffices to simplify the expression for $\vartheta(d)$,

$$\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{d-1}}{d} \leq \frac{1}{\vartheta(d)} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{d}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{d}{4}+1\right)} \leq \frac{2}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{d+1}}.$$

$\vartheta(2) = \frac{\pi}{2}$, $\vartheta(4) = 2$, $\vartheta(6) = \frac{3\pi}{4}$, $\vartheta(8) = \frac{8}{3}$, $\vartheta(10) = \frac{15\pi}{16}$, $\vartheta(12) = \frac{16}{5}$. □

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nc Computer Algebra Systems

Advertisement

There are several computer algebra packages available to aid with computations in a free setting.

(1) `NCAlgebra` running under Mathematica

http://math.ucsd.edu/~ncalg

(2) `NCSOStools` running under Matlab

http://ncsostools.fis.unm.si

The former is more universal in that it implements manipulation with noncommutative variables, including nc rationals, and several algorithms pertaining to convexity.

The latter is focused on nc positivity and numerics.

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Take-home messages

(1) Linear Matrix Inequalities (LMIs) are everywhere:

- "Borel measure" gets about 108 000 hits on google
- "Fock space" gets about 185 000 hits on google
- "linear matrix inequalities" gets about 216 000 hits on google

(2) Free LMI domination is equivalent to complete positivity:

$$\mathcal{D}_L \subseteq \mathcal{D}_{\tilde{L}} \quad \Longleftrightarrow \quad \tilde{L}(x) = V^T \big(I_\mu \otimes L(x)\big) V.$$

(3) Mutual LMI domination is equivalent to unitary equivalence:

$$\mathcal{D}_L = \mathcal{D}_{\tilde{L}} \quad \Longleftrightarrow \quad \tilde{L}(x) = U^T L(x) U, \ \ U \text{ unitary}.$$

(4) Dilation theory tells us how good the free relaxation is:

$$[-1,1]^s = \mathcal{D}_{\mathcal{C}_1}(1) \subseteq \mathcal{D}_L(1) \quad \Rightarrow \quad \frac{1}{\vartheta(d)} \mathcal{D}_{\mathcal{C}_1} \subseteq \mathcal{D}_L.$$

Take-home messages

(1) Linear Matrix Inequalities (LMIs) are everywhere:

- “Borel measure” gets about 25 000 hits on startpage
- “Fock space” gets about 35 000 hits on startpage
- “linear matrix inequalities” gets about 40 000 hits on startpage

(2) Free LMI domination is equivalent to complete positivity:

$$\tilde{L} \succeq 0 \text{ on } \mathcal{D}_L \quad \Longleftrightarrow \quad \tilde{L}(x) = \sum_j V_j^T L(x) V_j.$$

(3) Mutual LMI domination is equivalent to unitary equivalence:

$$\mathcal{D}_L = \mathcal{D}_{\tilde{L}} \quad \Longleftrightarrow \quad \tilde{L}(x) = U^T L(x) U, \ U \text{ unitary}.$$

(4) Dilation theory tells us how good the free relaxation is:

$$[-1,1]^s = \mathcal{D}_{\mathcal{C}_1}(1) \subseteq \mathcal{D}_L(1) \quad \Rightarrow \quad \frac{1}{\vartheta(d)} \mathcal{D}_{\mathcal{C}_1} \subseteq \mathcal{D}_L.$$