C^*-envelopes of Semicrossed Products

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Dynamical Systems

Let A is a C*-algebra and let P be an abelian semigroup acting A by completely contractive endomorphisms α:

$$\alpha_s : A \to A$$
$$\alpha_s \circ \alpha_t = \alpha_{s+t} \text{ for all } s, t \in P.$$

We call (A, α, P) a C*-dynamical system.

Covariant Pairs

Let (A, α, P) be a dynamical system. A covariant pair for the system (π, T) consists of

1. $T : P \to B(H)$, a contractive/isometric representation of P;
2. $\pi : A \to B(H)$, a completely contractive representation of A;
3. $\pi(a)T_s = T_s\pi(\alpha_s(a))$ for all $s \in P$, $a \in A$.

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Example (Fock Representation)

Let $\pi : A \to B(H)$ be a completely contractive representation of A. Let $K = H \otimes \ell^2(P)$. Define a representation of P on K by:

$$V_s(h \otimes e_t) = h \otimes e_{s+t};$$

and define a representation $\tilde{\pi}$ of A on K by

$$\tilde{\pi}(a)(h \otimes e_s) = \pi(\alpha_s(a))h \otimes e_s.$$

Then $(\tilde{\pi}, V)$ is a covariant pair for (A, α, P).

Semicrossed Products

Let (A, α, P) be a dynamical system. For the algebra $c_{00}(P, \alpha, A)$ of formal polynomials

$$\sum_{s \in P \text{ finite}} a_s \otimes e_s, \; a_s \in A,$$

with product

$$(a \otimes e_s)(b \otimes e_t) = a\alpha_t(b) \otimes e_{s+t}.$$

We define the semicrossed product of A by P, $A \times_\alpha P$, to be the completion of $c_{00}(A, \alpha, P)$ with respect to the seminorm

$$\| \sum_{s \in P} a_s \otimes e_s \| = \sup_{(\pi, T) \text{ contractive}} \| \sum_{s \in P} T_s \pi(a_s) \|.$$

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$$\|\sum_{s \in P} a_s \otimes e_s\| = \sup_{(\pi,T) \text{ contractive}} \|\sum_{s \in P} T_s \pi(a_s)\|.$$

Similarly, we define the isometric semicrossed product $A \times_\alpha^{\mathrm{is}} P$ using only isometric covariant pairs.

Some dilation results

In some circumstances it is known that $A \times_\alpha P$ and $A \times_\alpha^{\mathrm{is}} P$ are the same:

Theorem

1. *(Muhly-Solel (Sz.-Nagy))*

$$A \times_\alpha^{\mathrm{is}} \mathbb{Z}_+ \simeq A \times_\alpha \mathbb{Z}_+;$$

2. *(Ling-Muhly; Solel; DFK (Ando))*

$$A \times_\alpha^{\mathrm{is}} \mathbb{Z}_+^2 \simeq A \times_\alpha \mathbb{Z}_+^2.$$

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Note: It is not true for $\mathbb{Z}_+^3$ or beyond, since we do not have a 3-variable dilations or a von Neumann inequality (Parrott; Varopolous).

Recall the definition of the C^*-envelope

Let $\mathcal{A}$ be a unital (non-self-adjoint) operator algebra. If $\mathcal{C}$ is a C^*-algebra and

$$j : \mathcal{A} \to \mathcal{C}$$

is a completely isometric isomorphic embedding of $\mathcal{A}$ into $\mathcal{C}$ such that

$$\mathcal{C} = \mathrm{C}^*(j(\mathcal{A}))$$

then $\mathcal{C}$ is a C^*-cover of $\mathcal{A}$.
Note: C^*-covers are not unique.

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Definition (Arveson)

$\mathrm{C}^*_{\mathrm{env}}(A)$ is the smallest C^*-cover of $\mathcal{A}$. That is, if $\mathcal{C}$ is another C^*-cover for $\mathcal{A}$ then $\mathrm{C}^*_{\mathrm{env}}(A)$ is a quotient of $\mathcal{C}$, and the quotient map is completely isometrically isomorphic on the image of $\mathcal{A}$.

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Dritschell-McCullough (2005): Every completely contractive representation

$$\sigma : \mathcal{A} \to B(H)$$

can be dilated to a maximal representation ρ.
Then $\mathrm{C}^*_{\mathsf{env}}(\mathcal{A}) \simeq \mathrm{C}^*(\rho(\mathcal{A}))$.

C*-envelopes for semicrossed products over $\mathbb{Z}_+$ and $\mathbb{Z}_+^2$

Theorem

Let A be an operator algebra and let α be a completely isometric automorphism on a C^*-algebra A. Then (Muhly-Solel)

$$\mathrm{C}^*_{env}(A \times_\alpha \mathbb{Z}_+) \simeq A \rtimes_\alpha \mathbb{Z}.$$

More generally (Kakariadis-Katsoulis), if α is not an automorphism then $\mathrm{C}^*_{env}(A \times_\alpha \mathbb{Z}_+)$ is a full corner of group crossed product C^*-algebra $\tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z}$.

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Others who have calculated C*-envelopes of semicrossed products over $\mathbb{Z}_+$ include: Peters; Kakariadis-Katsoulis; Kakariadis; Davidson-Katsoulis.

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Abelian Lattice-Ordered Groups

Let G be a partially ordered abelian group. Suppose further that G is lattice-ordered with this ordering (i.e. every pair $g, h \in G$ has a least upper bound $g \vee h$ and a least lower bound $g \wedge h$). Let P be the positive cone in G:

$$P = \{g \in G \colon g \geq 0\}.$$

We will denote a lattice-ordered abelian group by (G, P).

Nica-Covariant representations

We don't want just any old representation of a semigroup, we want our representations to acknowledge the lattice structure.

Definition

Let (G, P) be a lattice-ordered abelian group. A contractive representation $T : P \to B(H)$ is Nica-covariant if

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Let V be an isometric Nica-covariant representation of P. Let $Q_s = V_s V_s^*$ for each $s \in P$. Then

$$Q_s Q_t = Q_{s \vee t} \text{ for all } s, t \in P.$$

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Note: A Nica-covariant representation of $\mathbb{Z}_+^k$ is given by k doubly-commuting contractions:

$$T_i T_j = T_j T_i \text{ and } T_i T_j^* = T_j^* T_i \text{ when } i \neq j.$$

Dilations of Nica-covariant representations

Theorem (Li 2015)

Let T be a contractive Nica-covariant representation of the positive cone P of a lattice-ordered group G. Then T dilates to an isometric Nica-covariant representation of P. **Note:** *Li does not require that G is abelian.*

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Definition (Nica-covariant semicrossed product)

Given a dynamical system (A, α, P), where (G, P) is a lattice-ordered abelian group, we define

$$A \times_{\alpha}^{\mathrm{nc}} P$$

to be the semicrossed product over all Nica-covariant covariant pairs.

Reminder of what we are up to..

Let (A, α, P) be a dynamical system, where A is a C*-algebra and (G, P) is a lattice-ordered abelian group.

1. We will not assume that each α_s is an automorphism ;
2. We will assume that each α_s is injective (for now).

We want to calculate $\mathrm{C}^*_{\mathsf{env}}(A \times_\alpha^{\mathrm{nc}} P)$, and relate it to a crossed product C*-algebra by the group G.

Automorphic Extensions

We can embed (A, α, P) into a group dynamical system $(\widetilde{A}, \widetilde{\alpha}, G)$.
To do this we us a (common) direct limit technique developed by Peters, Laca,...
Let $A_s = A$ for each $s \in P$. For $s \leq t$ define

$$\alpha_s^t : A_s \to A_t$$
$$\alpha_s^t = \alpha_{t-s}.$$

And define $\widetilde{A} = \lim_{\to} A_s$ to be the direct limit.
The endomorphisms α on A induce automorphisms $\widetilde{\alpha}$ on $\widetilde{A}$.

The C*-envelope of Nica-covariant Semicrossed Products

Theorem (Davidson-F.-Kakariadis)

Let (A, α, P) be a dynamical system, where A is a unital C-algebra, (G, P) is a lattice-ordered abelian group and each α_s is injective. Then*

$$\mathrm{C}^*_{env}(A \times_\alpha^{\mathrm{nc}} P) \simeq \widetilde{A} \rtimes_{\widetilde{\alpha}} G.$$

Removing Injectivity...

Restricting to $\mathbb{Z}_+^k$ we can remove the injectivity condition on $(A, \alpha, \mathbb{Z}_+^k)$. We want to embed $(A, \alpha, \mathbb{Z}_+^k)$ into an injective dynamical system $(B, \beta, \mathbb{Z}_+^k)$ (adding tails technique).

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Let

$$I_n = \bigcap_{n \wedge m = 0} \alpha_m^{-1} \left((\bigcap_{p \leq n} \ker \alpha_p)^\perp \right)$$

This is the largest ideal of A invariant under α_m when $m \wedge n = 0$.
Let $B_n = A/I_n$ and let

$$B = \sum_{n \in \mathbb{Z}_+^k}^{\oplus} B_n.$$

Note $A = B_0$.
Define β by

$$\beta_i(a_n)_n = (\alpha_i(a_n))_{i \wedge n = 0} + (a_{n-i})_{i \leq n}.$$

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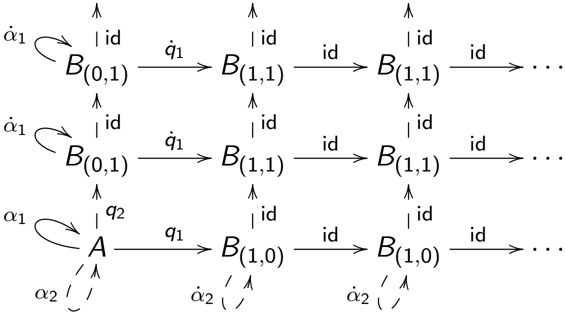

The C*-envelope

Theorem (Davidson-F.-Kakariadis)

Let $(A, \alpha, \mathbb{Z}_+^k)$ be a dynamical system where A is a unital C-algebra. Let $(B, \beta, \mathbb{Z}_+^k)$ be the injective dilation of $(A, \alpha, \mathbb{Z}_+^k)$. Then $\mathrm{C}^*_{env}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^k)$ is a full corner of $\widetilde{B} \rtimes_{\widetilde{\beta}} \mathbb{Z}^k$.*

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Using a gauge-invariant uniqueness theorem we can show that $\mathrm{C}^*_{\mathrm{env}}(A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n)$ is a Cuntz-Nica-Pimsner algebra of a product system.

Theorem

The C-envelope of $A \times_\alpha^{\mathrm{nc}} \mathbb{Z}_+^n$ is the universal algebra generated by Nica-covariant isometric pairs (π, V) such that*

$$\pi(a) \cdot \prod_{i \leq n} (I - V_i V_i^*) = 0 \text{ for all } a \in I_n.$$

Simplicity and Minimality

Similar to group crossed products, we can relate minimality of a dynamical system, to simplicity of a C*-algebra.

Definition

A C*-dynamical system (A, α, P) is minimal if A does not contain any non-trivial α-invariant ideals.

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Theorem (Davidson-F.-Kakariadis)

Let (G, P) be a lattice-ordered abelian group and (A, α, P) be an injective C-dynamical system. If $\mathrm{C}^*_{env}(A \times_\alpha^{\mathrm{nc}} P)$ is simple then (A, α, P) is minimal.*

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The converse is false in general (even for $P = \mathbb{Z}_+$). However, for classical systems we can say more.

Simplicity and Minimality for Classical Systems

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Theorem (Davidson-F.-Kakariadis)

Let (X, σ, P) be a surjective classical system over a lattice-ordered abelian group (G, P). Then the following are equivalent:

1. (X, σ, P) *is minimal and* $\sigma_s \neq \sigma_t$ *for all* $s, t \in P$, $s \neq t$.
2. *The C*-envelope of* $C(X) \times_\sigma^{\mathrm{nc}} P$ *is simple.*

Condition 1 above ensures that we get the necessary topological freeness condition.

Non-C*-dynamical systems

Theorem (Kakariadis-Katsoulis)

Let $(A, \alpha, \mathbb{Z}_+)$ be a dynamical system, with A not-necessarily a C*-algebra and α a completely isometric automorphism. Then

$$\mathrm{C}^*_{env}(A \times_\alpha^{\mathrm{is}} \mathbb{Z}_+) \simeq \mathrm{C}^*_{env}(A) \rtimes_\alpha \mathbb{Z}.$$

More generally if α is a completely contractive $*$-endomorphism which extends to $\mathrm{C}^*_{env}(A)$ then

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In certain cases it is known when $A \times_\alpha^{\mathrm{is}} \mathbb{Z}_+ \simeq A \times_\alpha \mathbb{Z}_+$, e.g. for $A = A(\mathbb{D})$, $A = \mathcal{A}_n$ (Davidson-Katsoulis) and $A = \mathcal{T}_X^+$ (Kakariadis-Katsoulis).

Other semigroups

Theorem (F.)

Let P be the positive cone of a direct sum of subgroups of $\mathbb{R}$, G. Let (A, α, P) be a dynamical system with α a completely isometric automorphisms. Then

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It it known when the Nica-covariant semicrossed product and the isometric Nica-covariant semicrossed coincide.