Essential normality of principal submodules of the Drury-Arveson module

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Plan of the Talk

1. Introduction
2. Recent Results
3. General description of the method

INTRODUCTION

Basic setting and notation:

$\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$, the unit ball in $\mathbf{C}^n$.

$S = \{z \in \mathbf{C}^n : |z| = 1\}$, the unit sphere in $\mathbf{C}^n$.

Standing Assumption: $n \geq 2$.

$dv =$ the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$.

$d\sigma =$ the spherical measure on S with the normalization $\sigma(S) = 1$.

$L^2_a(\mathbf{B}, dv)$: the Bergman space on $\mathbf{B}$.

$H^2(S)$: the Hardy space, which is a subspace of $L^2(S, d\sigma)$.

H^2_n : the Drury-Arveson space.

REPRODUCING-KERNEL HILBERT SPACES

Match spaces with reproducing kernels:

$$L_a^2(\mathbf{B}, dv) \quad : \quad \frac{1}{(1 - \langle \zeta, z \rangle)^{n+1}}$$

$$H^2(S) \quad : \quad \frac{1}{(1 - \langle \zeta, z \rangle)^{n}},$$

$$H_n^2 \quad : \quad \frac{1}{1 - \langle \zeta, z \rangle}.$$

In general the smaller the power of the denominator, the more difficult it is to deal with the space and the operators thereon.

Hilbert modules

These three spaces are all Hilbert modules over $\mathbf{C}[z_1, \ldots, z_n]$ under the identification of each $f \in \mathbf{C}[z_1, \ldots, z_n]$ with the multiplication operator M_f.

A submodule is a closed linear subspace $\mathcal{M}$ that is invariant under $M_{z_1}, \ldots, M_{z_n}$.

Each submodule $\mathcal{M}$ gives rise to the restricted module operators

$$Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}, \qquad j = 1, \ldots, n.$$

p-ESSENTIALLY NORMAL SUBMODULES

Suppose that $1 \leq p < \infty$.

A submodule $\mathcal{M}$ is said to be p-essentially normal if the commutators

$$[Z^*_{\mathcal{M},i}, Z_{\mathcal{M},j}], \qquad 1 \leq i, j \leq n,$$

all belong to the Schatten class $\mathcal{C}_p$.

Recall that an operator A belongs to $\mathcal{C}_p$ if and only if

$$\|A\|_p = \left\{ \mathrm{tr} \left((A^* A)^{p/2} \right) \right\}^{1/2} < \infty.$$

Arveson's Conjecture

Arveson's Conjecture. Every graded submodule $\mathcal{M}$ of H_n^2 is p-essentially normal for $p > n$.

Graded: $\mathcal{M}$ admits an orthogonal decomposition in terms of degree.

Many people have worked on this and related problems.

Partial list of names: Arveson, Douglas, Engliš, Eschmeier, Guo, Kennedy, Shalit, Wang.

The emphasis of Arveson's original problem is on graded submodules. Those are submodules generated by homogeneous polynomials.

In a quite unexpected development early 2011, Douglas and Wang proved

Theorem. If $[q]$ is the submodule of the Bergman module $L^2_a(\mathbf{B}, dv)$ generated by any $q \in \mathbf{C}[z_1, \dots, z_n]$, then $[q]$ is p-essentially normal for $p > n$.

This is an unconditional result in the sense that no assumption is made about the polynomial $q \in \mathbf{C}[z_1, \dots, z_n]$. This sets a very high standard for all subsequent investigations. More important, this signals the beginning of a new phase of investigations where one moves away from *degree-related* assumptions such as homogeneity.

GEOMETRIC VERSION

The newest development in this line of investigations are two noticeable papers on the geometric version of the essential normality problem:

"Geometric Arveson-Douglas conjecture" by Engliš and Eschmeier.

"An analytic Grothendieck Riemann Roch theorem" by Douglas, Tang and Yu.

The emphasis of these papers are on the geometric nature of the problem.

Recent Results

Inspired by the Douglas-Wang paper, we decided to take a look at the essential normality of principal submodules of the Hardy module and the Drury-Arveson module.

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The key realization is to treat the Bergman space, Hardy space and Drury-Arveson space in a unified way.

That is, these three spaces are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on $\mathbf{B}$ parametrized by a real-valued parameter (weight) $-n \leq t < \infty$.

In fact, the spaces corresponding to the values $t \in \mathbf{Z}_+$ were used in an essential way in the proofs in the Douglas-Wang paper.

Let us introduce the whole family of spaces.

A family of RKHS $\mathcal{H}^{(t)}$

For each real number $-n \le t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_n]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)}$$

if $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$.

One can think of the parameter t as the "weight" of the space, although t can be negative.

We have

$$\mathcal{H}^{(0)} = L_a^2(\mathbf{B}, dv), \qquad \text{the Bergman space,}$$

$$\mathcal{H}^{(-1)} = H^2(S), \qquad \text{the Hardy space,}$$

$$\mathcal{H}^{(-n)} = H_n^2, \qquad \text{the Drury-Arveson space.}$$

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The Bergman space $\mathcal{H}^{(0)}$ can be viewed as a benchmark, against which the other spaces in the family should be compared.

UNCONDITIONAL RESULT

Theorem 1.

Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \ldots, z_n]$. Then for each real number $-3 < t < \infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every $p > n$.

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Corollary. ($t = -1$) The submodule of the Hardy module $H^2(S)$ generated by any $q \in \mathbf{C}[z_1, \ldots, z_n]$ is p-essentially normal for every $p > n$.

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If we apply this to the case $t = -2$, we obtain the first non-trivial Drury-Arveson space case:

Corollary. ($n = 2$.) The submodule of H_2^2 generated by any $q \in \mathbf{C}[z_1, z_2]$ is p-essentially normal for every $p > 2$.

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In April 2011, we were able to extend our result to the case $-3 < t \leq -2$. But since this was obtained after our February 2011 submission, it has not yet appeared in publication. In fact, the result for the range $-3 < t \leq -2$ has just been submitted, along with a partial result for the case $t = -3$, , which will be the emphasis of this talk.

At the time we obtained our result for the weight range $-3 < t \leq -2$ (April 2011), we thought that the lower limit $t > -3$ represented the end of the road for this particular method.

The method that both Douglas-Wang and we used can be best described as the "slice method". We thought that "slice method" no longer worked for the case $t = -3$.

In other words, for the Drury-Arveson space in 3 varibles, H^2_3, we thought that new method was needed. (The issue here is not n, the number of variables, but rather the weight, t.)

But in the last few years, we never gave up on the case $t = -3$, and the effort paid off, at least partially.

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like to show that for every $q \in \mathbf{C}[z_1, \ldots, z_n]$, the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is essentially normal. This goal we have NOT achieved yet.

Instead, we are able to show that there is a substantial subclass $\mathcal{G}_n$ of $\mathbf{C}[z_1, \ldots, z_n]$ such that for every $q \in \mathcal{G}_n$, the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is essentially normal.

An important feature of the class $\mathcal{G}_n$ is that its membership is stable under small perturbation, in a sense to be made clear later.

To tackle the case $t = -3$, we need to consider the zero locus of q.

Given any $q \in \mathbf{C}[z_1, \ldots, z_n]$, we write

$$\mathcal{Z}(q) = \{z \in \mathbf{C}^n : q(z) = 0\}.$$

Write $\partial_1, \ldots, \partial_n$ for the differentiations with respect to the complex variables $z_1, \ldots, z_n$.

Recall that the n-variable radial derivative is given by the formula

$$R = z_1\partial_1 + \cdots + z_n\partial_n.$$

Definition. Let $\mathcal{G}_n$ be the collection of polynomials $q \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the following two conditions:

(a) The radial derivative Rq does not vanish on the set $\mathcal{Z}(q) \cap S$.

(b) The zero locus $\mathcal{Z}(q)$ intersects the unit sphere S transversely.

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(b) The zero locus $\mathcal{Z}(q)$ intersects the unit sphere S transversely.

Conditions (a), (b) above are inspired by Assumption 1.1 in the Douglas-Tang-Yu paper mentioned earlier.

Note that condition (a) implies that the analytic gradient $\partial q = (\partial_1 q, \dots, \partial_n q)$ does not vanish on the set $\mathcal{Z}(q) \cap S$, which ensures that (b) makes sense. At every point in S, the (real) co-dimension of the tangent space to S is 1. Thus condition (b) is simply equivalent to the condition that if $\xi \in \mathcal{Z}(q) \cap S$, then the tangent space to $\mathcal{Z}(q)$ at ξ is not contained in the tangent space to S at ξ.

It is easy to see that the membership $q \in \mathcal{G}_n$ is equivalent to the condition that the strict inequality

$$0 < |(Rq)(\xi)| < |(\partial q)(\xi)|$$

holds for every $\xi \in S \cap \mathcal{Z}(q)$, where $\partial q = (\partial_1 q, \ldots, \partial_n q)$, and $|(\partial q)(\xi)|$ is the Euclidian length of the vector $(\partial q)(\xi)$.

PARTIAL RESULT FOR $t = -3$

Here is what we can prove in the case $t = -3$:

Theorem 2

If $q \in \mathcal{G}_n$, $n \geq 3$, then the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is p-essentially normal for every $p > n$.

In the case $n = 3$, we have $\mathcal{H}^{(-3)} = H^2_3$, the Drury-Arveson space in three variables. Therefore the above implies

Corollary

If $q \in \mathcal{G}_3$, then the submodule $[q]$ of H^2_3 is p-essentially normal for every $p > 3$.

General description of the method

For the weight range $-3 \le t \le -2$, the central issue in the proof essential normality revolves around just one single inequality. The best way to explain this is to introduce

Definition. Suppose that $-n \le t < \infty$ and $0 \le \epsilon < 1$. Then a polynomial $q \in \mathbf{C}[z_1, \ldots, z_n]$ is said to be in the class $\mathcal{P}_n(t; \epsilon)$ if there is a $0 < C = C(q) < \infty$ such that

$$\|fRq\|_{t+3} \le C\|qf\|_{t+1-\epsilon}$$

for every $f \in \mathbf{C}[z_1, \ldots, z_n]$.

The ϵ above gives us a bit of leeway in the proofs. The class $\mathcal{P}_n(t; \epsilon)$ is a stepping stone on the way to essential normality:

Proposition 1. *Suppose that* $t \geq -3$ *and that* $0 \leq \epsilon < 1$. *Let* $q \in \mathcal{P}_n(t;\epsilon)$. *Then the submodule* $[q]^{(t)}$ *of* $\mathcal{H}^{(t)}$ *is essentially normal. More precisely, the submodule operators*

$$Z_{q,j}^{(t)} = M_{z_j}|[q]^{(t)}, \quad 1 \leq j \leq n,$$

have the property $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}] \in \mathcal{C}^+_{n/(1-\epsilon)}$ *for all* $j, i \in \{1, \ldots, n\}$.

Recall that for $1 \leq p < \infty$, $\mathcal{C}_p^+$ is the norm ideal consisting of operators T satisfying the condition

$$\|T\|_p^+ = \sup_{k \geq 1} \frac{s_1(T) + s_2(T) + \cdots + s_k(T)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}} < \infty.$$

Also, if $1 \leq p < r < \infty$, then the ideal $\mathcal{C}_p^+$ is contained in the Schatten class $\mathcal{C}_r$

Proposition 2. *For each* $-3 < t \leq -2$ *we have* $\mathcal{P}_n(t;0) = \mathbf{C}[z_1, \ldots, z_n]$.

Propositions 1 and 2 immediately give us the result that for every $q \in \mathbf{C}[z_1, \ldots, z_n]$, if $-3 < t \leq -2$, then the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is essentially normal.

In view of the previous page, we obviously would like to show that

$$\mathcal{P}_n(-3; 0) = \mathbf{C}[z_1, \ldots, z_n],$$

or at least that

$$\mathcal{P}_n(-3; \epsilon) = \mathbf{C}[z_1, \ldots, z_n]$$

for $0 < \epsilon < 1$, which would settle the case $t = -3$ completely. But we are not able to prove either of these at the moment.

Instead, this is the best we can do for the moment:

Proposition 3. *For every pair of* $n \geq 3$ *and* $0 < \epsilon < 1/2$ *we have* $\mathcal{G}_n \subset \mathcal{P}_n(-3; \epsilon)$.

This and Proposition 1 together give us Theorem 2, our essential normality result for the case $t = -3$.

Next let us explain where the difficulties are for the case $t = -3$, particularly in comparison with the case $-3 < t \leq -2$.

The method we use can be best described as the slice method, which first appeared in the Douglas-Wang paper and which is based on the formula

$$\int G dv = n \int \left(\int G_\xi(z) |z|^{2n-2} dA(z) \right) d\sigma(\xi).$$

For each $\xi \in S$, the one-variable function $G_\xi(z) = G(z\xi)$ is called a "slice" of G, hence the term slice method.

The obvious advantage of the method is that it reduces multi-variable estimates to estimates on the unit disc.

Limitation: it is possible that an inequality may hold in the case of two or more variables, but its one-variable counter part actually fails. In such an event, one may have to do a "salvage operation".

For each $-1 < r \leq 1$ and each one-variable polynomial f, define

$$\mathcal{N}_r(f) = |f(0)|^2 + \int |(\mathcal{R}f)(z)|^2 (1 - |z|^2)^r dA(z).$$

Here $\mathcal{R}$ denotes the one-variable radial derivative $z(d/dz)$ on the unit disc D. The r above is a "shift" of the weight t in question.

Proposition 4. *Suppose that* $0 < r \leq 1$. *Then there is a constant* $0 < C(r) < \infty$ *such that if* g *and* f *are one-variable polynomials and if* $\deg(g) = K \geq 1$, *then*

$$\int |(\partial g)(z) f(z)|^2 (1 - |z|^2)^r dA(z) \leq C(r) K^2 \mathcal{N}_r(gf).$$

This is the one-variable estimate on which the essential normality result for the range $-3 < t \leq -2$ depends.

To obtain the desired essential normality in the case $t = -3$ using the same method, one would have to prove Proposition 4 for the case $r = 0$. That is, one would need an inequality of the form

$$(**) \qquad \int |(\partial g)(z) f(z)|^2 dA(z) \leq C\mathcal{N}_0(gf),$$

where C is independent of f. But if one simply tries $g(z) = 1 - z$, one sees that $(**)$ in general fails.

At first and then for quite a while, we thought that this failure meant that the case $t = -3$ was hopeless. But eventually we took a closer look at how $(**)$ fails, and we realized that it is still possible to show that $\mathcal{P}_n(-3; \epsilon)$ contains a substantial subset of $\mathbf{C}[z_1, \ldots, z_n]$.

Indeed a careful analysis shows that the example $1 - z$ is already the worst case scenario for $(**)$, namely g has a zero on the unit circle $\mathbf{T}$. Here g represents the slices q_ξ, $\xi \in S$, of the n-variable polynomial q under consideration. Thus our analysis tells us that if the circle

$$\{\tau\xi : \tau \in \mathbf{T}\}$$

runs through the zero locus $\mathcal{Z}(q)$, then q_ξ is a bad slice for q. But fortunately, there are not too many such bad slices for each $q \in \mathcal{G}_n$, and the other slices of such a q are all "salvageable". This is the idea behind the proof of Theorem 2. But it takes quite a bit of work to bring this idea to fruition.

Note that by the product rule for the radial derivative R, the set $\mathcal{P}_n(t;\epsilon)$ is multiplicative for all $-n \le t < \infty$ and $0 \le \epsilon < 1$. That is, for $q_1, \ldots, q_k \in \mathcal{P}_n(t;\epsilon)$, $k \ge 1$, we have $q_1 \cdots q_k \in \mathcal{P}_n(t;\epsilon)$. Of course, this fact is not significant in cases where we know that the equality $\mathcal{P}_n(t;\epsilon) = \mathbf{C}[z_1, \ldots, z_n]$ holds. But in cases where we do not yet know this equality for a fact, the multiplicativity of $\mathcal{P}_n(t;\epsilon)$ becomes significant. Indeed using this multiplicativity we actually obtain

Corollary. *If $q_1, \ldots, q_k \in \mathcal{G}_n$, $n \ge 3$ and $k \ge 1$, then the submodule $[q_1 \cdots q_k]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is p-essentially normal for every $p > n$.*

The most prominent feature of $\mathcal{G}_n$ is that its membership is stable under small perturbation. To make this precise, we need to introduce a norm. For any function h that is analytic on an open set Ω containing the closed ball $\overline{\mathbf{B}}$, we define

$$\|h\|_{\#} = \max\left\{\max_{|z|\le 1}|h(z)|, \max_{|z|\le 1}|(\partial h)(z)|\right\}.$$

Proposition. *For each $q \in \mathcal{G}_n$, there is a $\rho > 0$ such that for every $h \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the condition $\|h\|_{\#} \le \rho$, we have $q + h \in \mathcal{G}_n$.*

Thanks for your attention!

Example 9.2. Let $a \in \mathbf{C}$ be such that $|a| = 1/2$. If $h \in \mathbf{C}[z_1, \ldots, z_n]$ satisfies the condition $\|h\|_\# \leq 1/8$, then the polynomial

$$q(z_1, \ldots, z_n) = z_1 - a + h(z_1, \ldots, z_n)$$

belongs to $\mathcal{G}_n$. Indeed if $\xi = (\xi_1, \ldots, \xi_n)$ belongs to $\mathcal{Z}(q) \cap S$, then we have $\xi_1 - a + h(\xi_1, \ldots, \xi_n) = 0$. Since $\|h\|_\# \leq 1/8$, this implies $3/8 \leq |\xi_1| \leq 5/8$, and consequently

$$|(Rq)(\xi_1, \ldots, \xi_n)| = |\xi_1 - (Rh)(\xi_1, \ldots, \xi_n)| \leq (5/8) + (1/8) = 3/4.$$

On the other hand, for every $\zeta \in S$ we have $|(\partial q)(\zeta)| \geq 1 - |(\partial h)(\zeta)| \geq 1 - (1/8) > 3/4$. Therefore $|(\partial q)(\xi)| > |(Rq)(\xi)|$ for every $\xi \in \mathcal{Z}(q) \cap S$. For $(\xi_1, \ldots, \xi_n) \in \mathcal{Z}(q) \cap S$, we also have

$$|(Rq)(\xi_1, \ldots, \xi_n)| \geq |\xi_1| - |(Rh)(\xi_1, \ldots, \xi_n)| \geq (3/8) - (1/8) > 0.$$

Hence $q \in \mathcal{G}_n$. Note that if there are $a_2, \ldots, a_n$ such that $(a, a_2, \ldots, a_n) \in \mathbf{B}$ and $h(a, a_2, \ldots, a_n) = 0$, then we also have $q(a, a_2, \ldots, a_n) = 0$.

Ingredients of the proofs

Definition. For a one-variable polynomial g of degree at least 1, we write

$$\Delta(g) = \inf\{|a - \tau| : g(a) = 0,\ \tau \in \mathbf{T}\}.$$

Proposition 5. *Suppose that* $0 < \epsilon < 1$. *Let* g *and* f *be one-variable polynomials. If the degree of* g *equals* $K \geq 1$ *and if* g *has no zeros on the unit circle* $\mathbf{T}$, *then*

$$\int |(\partial g)(z) f(z)|^2 dA(z) \leq \frac{C(\epsilon)}{(\Delta(g))^{\epsilon}} K^2 \mathcal{N}_0(gf),$$

Definition. Let $q \in \mathbf{C}[z_1, \ldots, z_n]$.
(1) Set $\mathcal{B}(q) = \{\tau\zeta : \zeta \in S \cap \mathcal{Z}(q), \tau \in \mathbf{T}\}$.
(2) For each $\xi \in S$, let $\Delta(q;\xi) = \inf\{|\tau\xi - \zeta| : \tau \in \mathbf{T}, \zeta \in \mathcal{Z}(q)\}$.
(3) For each $0 < \epsilon < 1$, let $\mu_{q;\epsilon}$ be the measure on S given by the formula

$$\mu_{q;\epsilon}(A) = \int_{A \setminus \mathcal{B}(q)} \frac{1}{(\Delta(q;\xi))^{\epsilon}} d\sigma(\xi)$$

for Borel sets $A \subset S$.

Note that $\Delta(q;\xi)$ is the distance between the circular slice $\{\tau\xi : \tau \in \mathbf{T}\}$ and the zero locus $\mathcal{Z}(q)$.

As the notation suggests, $\mathcal{B}(q)$ is the bad set for q.

Proposition 6. *For* $q \in \mathbf{C}[z_1, \ldots, z_n]$ *and* $0 < \epsilon < 1/2$*, we have* $q \in \mathcal{P}_n(-3; \epsilon)$ *whenever the following two conditions are satisfied:*
(1) $\sigma(\mathcal{B}(q)) = 0$.
(2) *There is a constant* C *such that*

$$\int \int |h(z\xi)|^2 dA(z) d\mu_{q;2\epsilon}(\xi) \leq C \int |h(\zeta)|^2 (1 - |\zeta|^2)^{-\epsilon} dv(\zeta)$$

for every $h \in \mathbf{C}[z_1, \ldots, z_n]$.

Obviously, this tells us how to proceed: show that every $q \in \mathcal{G}_n$ satisfies conditions (1) and (2) above. But this takes quite a few steps.

For each pair of $\xi \in S$ and $r > 0$, we denote

$$S(\xi, r) = \{x \in S : |x - \xi| < r\},$$

the intersection of the Euclidian ball $\{z \in \mathbf{C}^n : |z - \xi| < r\}$ with the unit sphere.

The key step is a distribution inequality:

Lemma 1. *For each $q \in \mathcal{G}_n$, there exist $r_0 > 0$ and $0 < C < \infty$ such that the inequality*

$$\sigma(\{w \in S(\xi, r) : \Delta(q; w) < \rho\}) \leq Cr^{2n-2}\rho$$

holds for all $\xi \in S$ and $0 < \rho \leq r < r_0$.

The idea for the proof of Lemma 1 is actually quite simple. Namely, for sufficiently small $0 < \rho \leq r$, we can cover the set $\{w \in S(\xi, r) : \Delta(q; w) < \rho\}$ with a family of balls $\{E_\nu : \nu \in \mathcal{N}\}$ in the Euclidian metric, where the radius of each E_ν is on the order of ρ and where the cardinality of $\mathcal{N}$ is on the order of $(r/\rho)^{2n-2}$. Then, since $\sigma(S \cap E_\nu)$ is on the order of ρ^{2n-1}, the desired estimate follows.

But the details of the proof are unfortunately quite complicated.

Two consequences of Lemma 1:

Proposition 7. *If* $q \in \mathcal{G}_n$*, then* $\sigma(\mathcal{B}(q)) = 0$*. In other words, every* $q \in \mathcal{G}_n$ *satisfies condition* (1) *in Proposition* 6.

Proposition 8. *Let* $q \in \mathcal{G}_n$*. Then for every* $0 < \epsilon < 1$*, there is a constant* C *such that* $\mu_{q;\epsilon}(S(\xi, r)) \le Cr^{2n-1-\epsilon}$ *for all* $\xi \in S$ *and* $r > 0$.

In other words, if $q \in \mathcal{G}_n$, then the growth rate of $\mu_{q;\epsilon}$ is worse than that of the spherical measure σ by at most ϵ.

Further implications:

Proposition 9. *Let μ be a Borel measure on S and suppose that $0 < \epsilon < 1/2$. If there is a constant C such that $\mu(S(\xi, r)) \leq Cr^{2n-1-2\epsilon}$ for all $\xi \in S$ and $r > 0$, then there is a constant C such that*

$$\int \int \left| \frac{1}{(1 - \langle z\xi, w \rangle)^{n+1-\epsilon}} \right|^2 dA(z) d\mu(\xi) \leq \frac{C}{(1 - |w|^2)^{n+1-\epsilon}}$$

for every $w \in \mathbf{B}$.

The above inequality is easily recognizable as a Carleson condition for the weighted bergman space $\mathcal{H}^{(-\epsilon)}$. Accordingly, one expects the consequent boundedness:

Proposition 10. *Let μ be a Borel measure on S and suppose that $0 < \epsilon < 1$. If there is a constant C such that*

$$\int\int \left| \frac{1}{(1 - \langle z\xi, w\rangle)^{n+1-\epsilon}} \right|^2 dA(z)d\mu(\xi) \le \frac{C}{(1 - |w|^2)^{n+1-\epsilon}}$$

for every $w \in \mathbf{B}$, then there is a constant C such that

$$\int\int |h(z\xi)|^2 dA(z)d\mu(\xi) \le C \int |h(\zeta)|^2 (1 - |\zeta|^2)^{-\epsilon} dv(\zeta)$$

for every $h \in \mathbf{C}[z_1, \ldots, z_n]$.

The combination of Propositions 8,9 and 10 tells us that each $q \in \mathcal{G}_n$ also satisfies condition (2) in Proposition 6. We have already seen that each $q \in \mathcal{G}_n$ satisfies condition (1). This gives us

Proposition 11. *For every pair of $n \geq 3$ and $0 < \epsilon < 1/2$ we have $\mathcal{G}_n \subset \mathcal{P}_n(-3; \epsilon)$.*

This and Proposition 1 together give us Theorem 2, our essential normality result in the case $t = -3$.

Thanks for your attention!