Current Density Impedance Imaging with Complete Electrode Model

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Outline

In motivation
- Hybrid methods in Inverse Problems
- Current density based EIT
- Acquiring the interior data

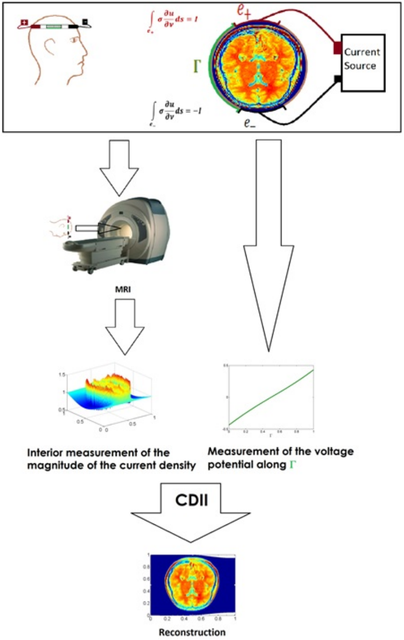
The forward problem
- The Complete Electrode Model

The Inverse Problem
- Characterization of non-uniqueness
- Phase retrieval
- Restoring uniqueness
- A numerical algorithm and experiment
- Conclusions

Coupled Physics Imaging Methods

Combine high contrast & high resolution

- Elastography: elastic waves & ultrasound/MRI $\Rightarrow$ stiffness
- Thermo/PhotoAcoustic: UV light & sound $\Rightarrow$ embedded acoustic sources
- AcoustoOptics: light & sound $\Rightarrow$ absorption and scattering
- **Coupled Physics Electrical Impedance Tomography**
  - Current density impedance imaging **CDII**: Joy& Nachman since 2002, Seo et al. 2002
  - MREIT (B_z-methods): Seo et al. since 2003
  - Ultrasound modulated EIT: Capdebosq et at. 2008, Bal et al. 2009
  - Impedance acoustic: Scherzer et al. 2009
  - Lorentz force driven EIT: Ammari et al. since 2013
  - ...



e+
$\int_{e_+} \sigma \frac{\partial u}{\partial \nu} ds = I$
Γ
Current Source
$\int_{e_-} \sigma \frac{\partial u}{\partial \nu} ds = -I$
e−
MRI
Interior measurement of the magnitude of the current density
Measurement of the voltage potential along Γ
CDII
Reconstruction

Current density tracing inside an object

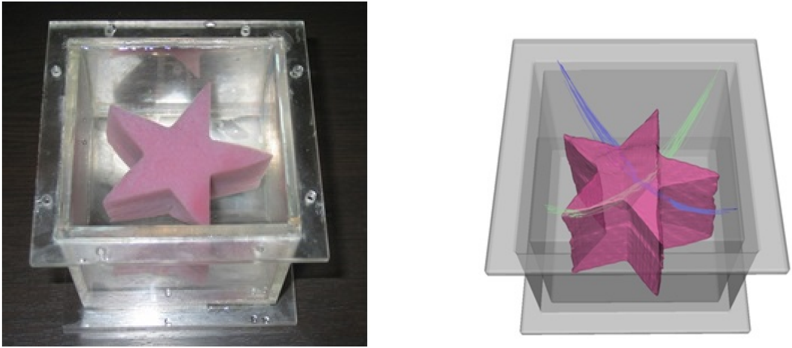


Figure : Courtesy: Joy's group, U Toronto

Magnetic resonance data: $M : \Omega \to \mathbb{C}$

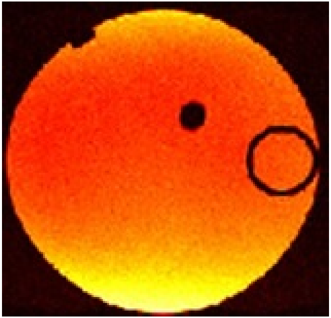


Figure : $M_{\pm}(x, y, z_0) = M(x, y, z_0) \exp(\pm i\gamma B_z(x, y, z_0) T + i\varphi_0)$

Aquiring the interior data

One MR scan ⇒ longitudinal component B_z (along gantry) of the magnetic field $\mathbf{B} = (B_x, B_y, B_z)$

$$B_z(x, y, z_0) = \frac{1}{2\gamma T} \mathbb{I}\text{m}\, log\left(\frac{M_+(x, y, z_0)}{M_-(x, y, z_0)}\right)$$

- MREIT (Seo at al. since 2003): Does B_z uniquely determine the electrical conductivity? In general, not known.
- CDII (Nachman et al since 2002, Seo (2002)) : + two rotation of the object

$$\Rightarrow \mathbf{B} \Rightarrow \mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

- Anisotropic case: Bal & Monard (2013), unique determination Hoell-Moradifam-Nachman (2014, within conformal class)

Today: the magnitude $|\mathbf{J}|$ is assumed known inside .

Outline

Complete Electrode Model (Somersalo-Cheney-Isaacson ’92)

$\Omega \subset \mathbb{R}^n$ bounded with Lipschitz boundary $\partial\Omega$,
$N+1$ electrodes: $e_k \subset \partial\Omega$, $k = 0, ..., N$,
$\epsilon \leq \mathbb{R}\mathrm{e}\{\sigma\} \leq 1/\epsilon$,
$\epsilon \leq \mathbb{R}\mathrm{e}\{z_k\} \leq 1/\epsilon$, $k = 0, 1, ..., N$,

$$\nabla \cdot \sigma \nabla u = 0, \quad \text{in } \Omega,$$

$$u + z_k \sigma \frac{\partial u}{\partial \nu} \equiv const = U_k \quad \text{on } e_k, \quad \text{for } k = 0, ..., N,$$

$$\int_{e_k} \sigma \frac{\partial u}{\partial \nu} ds = I_k, \quad \text{for } k = 0, ..., N,$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega \setminus \bigcup_{k=0}^{N} e_k,$$

Forward problem (CEM) is well posed

Based on Lax-Milgram lemma:
Theorem (Somersalo- Cheney- Isaacson '92) Provided

$$\sum_{k=0}^{N} I_k = 0,$$

there is a unique solution $\langle u(x), (U_0, ..., U_N) \rangle \in H^1(\Omega) \times \mathbb{C}^{N+1}$ up to a constant.

Normalization

Uniqueness up to a constant:
$\langle u(x) + c, (U_0 + c, ..., U_N + c) \rangle$ also a solution.

$$\nabla \cdot \sigma \nabla (u + c) = 0, \quad \text{in } \Omega,$$

$$(u + c) + z_k \sigma \frac{\partial (u + c)}{\partial \nu} \equiv const = U_k + c \quad \text{on } e_k, \ \text{ for } k = 0, ..., N,$$

$$\int_{e_k} \sigma \frac{\partial (u + c)}{\partial \nu} ds = I_k, \quad \text{for } k = 0, ..., N,$$

$$\frac{\partial (u + c)}{\partial \nu} = 0, \quad \text{on } \partial\Omega \setminus \bigcup_{k=0}^{N} e_k,$$

Normalization: fix a constant by seeking $\mathbf{U} = (U_0, ..., U_N)$ with $\sum_{k=0}^{N} U_k = 0$.

New properties in the real valued case

$$\sigma(x), z_0(x), ..., z_N(x) \in \mathbb{R}$$

$$\mathbf{U} \in \Pi := \{(U_0, ...U_N) \in \mathbb{R}^{N+1} : \sum_{k=0}^{N} U_k = 0\}$$

- Maximum Principle for CEM: The maximum and minimum of the voltage potential u occur on the electrodes.
- A Poicaré Inequality (not necessarily connected with CEM): $\exists C > 0$ dependent only on Ω and $e_k \subset \partial\Omega$ such that $\forall u \in H^1(\Omega)$ and $\forall \mathbf{U} = (U_0, ..., U_N) \in \Pi$:

$$\int_\Omega u^2 + \sum_{k=0}^{N} U_k^2 \leq C \left(\int_\Omega |\nabla u|^2 dx + \sum_{k=0}^{N} \int_{e_k} (u - U_k)^2 ds \right)$$

The Dirichlet principle for the CEM

Consider the functional

$$F_\sigma(u,\mathbf{U}) := \frac{1}{2}\int_{\partial\Omega}\sigma|\nabla u|^2 dx + \frac{1}{2}\sum_{k=0}^{N}\int_{e_k}\frac{1}{z_k}(u-U_k)^2 ds - \sum_{k=0}^{N} I_k U_k.$$

Recall Ω, Π, $e_k \subset \partial\Omega$, z_k, for $k = 0, ..., N$, σ, and

$$\sum_{k=0}^{N} I_k = 0 \qquad (\star)$$

Theorem(Nachman-T-Veras '14)
(i) Independently of $(\star)$:

$$\exists!\ (u,\mathbf{U}) = \text{argmin}_{H^1(\Omega)\times\Pi} F_\sigma$$

(ii) If $(\star)$ holds:

$$(u,\mathbf{U}) = \text{argmin}_{H^1(\Omega)\times\Pi} F_\sigma \Leftrightarrow (u,\mathbf{U}) \text{ solves CEM}$$

Formulation of an Inverse Problem

Given: Ω, $e_k \subset \partial\Omega$ with $z_k > 0$, and $I_1, ..., I_N$,
(then $I_0 := -\sum_{k=1}^{N} I_k$),
and $|\mathbf{J}| = \sigma|\nabla u|$ inside Ω,

Find σ.

Formulation of an Inverse Problem

Given: Ω, $e_k \subset \partial\Omega$ with $z_k > 0$, and I_k, ${}_{k=1,\dots,N}$ (then $I_0 := -\sum_{k=1}^N I_k$), and $|\mathbf{J}| = \sigma|\nabla u|$ inside,
Find σ**.**
Not possible:
$\Omega = (0,1) \times (0,1)$,
Top side: e_1 with $z_1 > 0$, inject $I_1 = 1$
Bottom side: e_0 with $z_0 = z_1 + 1$, extract $I_0 = -1$
Measure the magnitude $|\mathbf{J}| \equiv 1$ inside.
Arbitrary $\varphi : [0,1] \to [\varphi(0), \varphi(1)]$ increasing, Lipschitz with $\varphi(0) + \varphi(1) = 1$.
Then: $u_\varphi(x,y) := \varphi(y)$ voltage for $\sigma_\varphi(x,y) = 1/\varphi'(y)$.
Yet for all such φ,

$$\sigma_\varphi |\nabla u_\varphi| \equiv 1!$$

Generic non-uniqueness

Let $(u, U) \in H^1(\Omega) \times \Pi$ be the solution of CEM for some σ.
$\varphi \in Lip(u(\overline{\Omega}))$ be an increasing function of one variable,
$\varphi(t) = t + c_k$ whenever $t \in u(e_k)$, for each $k = 0, ..., N$, and
constants c_k satisfying $\sum_{k=0}^{N} c_k = 0$. Then

$$u_\varphi := \varphi \circ u \tag{1}$$

is a voltage potential for CEM with

$$\sigma_\varphi := \frac{\sigma}{\varphi' \circ u}, \tag{2}$$

and has the same interior data

$$\sigma|\nabla u| = \sigma_\varphi |\nabla u_\varphi|.$$

Characterization of Non-uniqueness

Theorem (Nachman-T-Veras '14) Recall assumptions on $\Omega \subset \mathbb{R}^d$ be bounded, connected $C^{1,\alpha}$, $e_k \subset \partial\Omega$, $z_k > 0$, I_k, $k = 0, ..., N$.
Let $(u, U), (v, V) \in H^1(\Omega) \times \Pi$, be the CEM solutions for unknown conductivities $\sigma, \tilde{\sigma} \in C^\alpha(\Omega)$ with

$$|\mathbf{J}| := \sigma|\nabla u| = \tilde{\sigma}|\nabla v| > 0 \ a.e. \text{in } \Omega.$$

Then $\exists \varphi \in C^1(u(\Omega))$, with $\varphi'(t) > 0$ a.e. in Ω, such that

$$v = \varphi \circ u, \quad \text{in } \Omega,$$

$$\tilde{\sigma} = \frac{\sigma}{\varphi' \circ u}, \quad \text{a.e. in } \Omega.$$

Moreover, for each $k = 0, ..., N$ and $t \in v(e_k)$,

$$\varphi(t) = t + (U_k - V_k).$$

Idea: reduction to a minimization problem

Inverse hybrid problem: Consider

$$G_{|\mathbf{J}|}(v, V) = \int_{\Omega} |\mathbf{J}|\, |\nabla v| dx + \frac{1}{2}\sum_{k=0}^{N}\int_{e_k}\frac{1}{z_k}(v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k,$$

- solutions of CEM are global minimizers of $G_{|\mathbf{J}|}$ over $H^1(\Omega) \times \Pi$.
- Geometry of the equipotential sets are uniquely determined! Contrast with Dirichlet

Contrast with functional in the forward model

$$F_{\sigma}(v, \mathbf{V}) := \frac{1}{2}\int_{\partial\Omega} \sigma |\nabla v|^2 dx + \frac{1}{2}\sum_{k=0}^{N}\int_{e_k}\frac{1}{z_k}(v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k.$$

Corollaries

- **Phase retrieval** (Nachman-T-Veras'14) Same hypotheses (recall).

$$|\mathbf{J}| = |\tilde{\mathbf{J}}| \Rightarrow \mathbf{J} = \tilde{\mathbf{J}}.$$

- There is uniqueness (and a reconstruction method) from the magnitudes of **two** currents via a local formula (Nachman et al., Lee 2004)
- The *J*-substitution algorithm via magnitudes of **two** currents (Seo et al 2002) converges to the unique solution.

Knowledge of the potential on a boundary curve joining the electrodes restores uniqueness

Theorem (Nachman-T-Veras '14) In addition to the hypotheses of the characterization theorem if

$$u|_\Gamma = \tilde{u}|_\Gamma + C,$$

for some C, and Γ a curve joining the electrodes, then

$$u = \tilde{u} + C \text{ in } \overline{\Omega},$$
$$\sigma = \tilde{\sigma} \text{ in } \Omega.$$

A minimization algorithm for G

$$G_{|\mathbf{J}|}(v, V) = \int_\Omega |\mathbf{J}| \, |\nabla v| dx + \sum_{k=0}^{N} \int_{e_k} \frac{1}{2z_k}(v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k,$$

Lemma Assume that $v \in H^1(\Omega)$ satisfies

$$\epsilon \leq \frac{a}{|\nabla v|} \leq \frac{1}{\epsilon},$$

for some $\epsilon > 0$, and let $(u, U) \in H^1(\Omega) \times \Pi$ be the unique solution for CEM with $\sigma := a/|\nabla v|$. Then

$$G_a(u, U) \leq G_a(v, V), \quad \text{for all } V \in \Pi.$$

Moreover, if equality holds then $(u, U) = (v, V)$.

A minimization algorithm

- With σ_n given: Solve CEM for the unique solution $(u_n, \mathbf{U}^n)$;
- If
$$\text{essinf}\|\nabla u_n - \nabla u_{n-1}\| > \delta \frac{\epsilon}{\text{essinf}|\mathbf{J}|},$$
update
$$\sigma_{n+1} := \min\left\{\max\left\{\frac{|\mathbf{J}|}{|\nabla u_n|}, \epsilon\right\}, \frac{1}{\epsilon}\right\}$$
and repeat;
- else STOP.

Enough for the phase retrieval:

$$\mathbf{J} \approx |\mathbf{J}| \frac{\nabla u_n}{|\nabla u_n|}$$

Using the voltage on Γ

Let n be the last iteration and set

$$\sigma_{n+1} := \frac{|\mathbf{J}|}{\nabla u_n}.$$

The Characterization Theorem

$$\Rightarrow u(x) \approx f(u_n(x)).$$

Read off the measured data on Γ to determine the scaling function $f : u(\Gamma) \to u_n(\Gamma)$.
Then

$$\sigma(x) \approx \frac{1}{f'(u_n(x))}\sigma_{n+1}(x).$$

Reconstruction results in a numerical experiment

$$1S/m \leq \sigma \leq 1.8S/m, \; -I_0 = I_1 = 3mA, \; z_0 = z_1 = 8.3m\Omega \cdot m^2$$

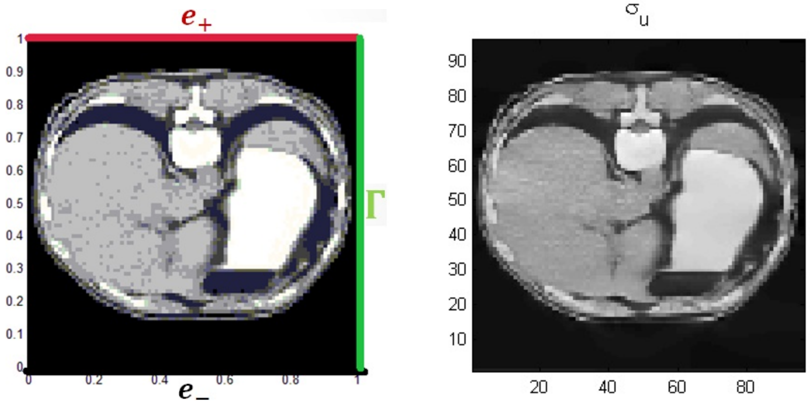


Figure : Exact conductivity (left) vs. reconstructed conductivity (right)

Voltage potential scaling along Γ

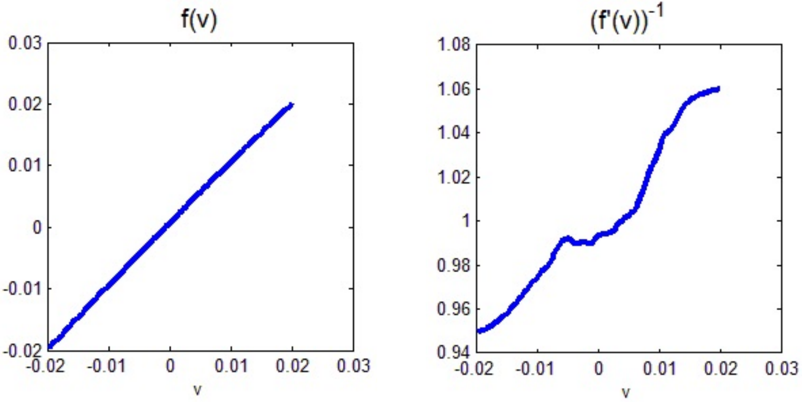


Figure : The scaling function f and its derivative.

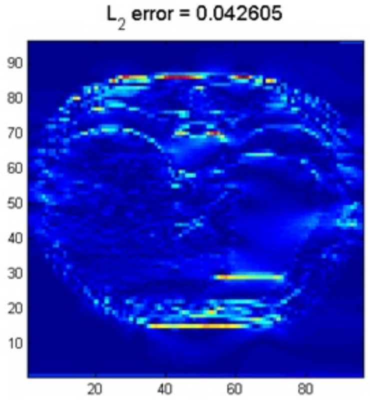


Figure : L^2-Error: Understood from the stability in the linearized case Kuchment&Steinhauer (2011), Bal (2012)

Some learnings

- in the more realistic CEM. the magnitude of one current density by itself cannot determine an isotropic conductivity
- the magnitude of two currents uniquely determine the conductivity (up to an additive constant)
- in the isotropic case: the phase of the current is uniquely determined from its magnitude (not known in the anisotropic case)
- knowledge of the voltage potential along a curve restores uniqueness
- the method is constructive

Thank you!