On Proper Data Sets for Elliptic Hybrid Inverse Problems

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Introduction

Impedance reconstruction (Calderón's Problem) from boundary measurement is very difficult.
In Ω bounded, smooth, finding γ from

$$
\begin{array}{rcl}
\Lambda_\gamma : H^{-1/2}(\partial\Omega) & \to & H^{1/2}(\partial\Omega) \\
\phi & \to & u|_{\partial\Omega} : \\
 & & -\mathrm{div}\,(\gamma\nabla u) = 0 \text{ in } \Omega, \\
 & & a\nabla u \cdot n = \phi \text{ on } \partial\Omega.
\end{array}
$$

- Need $\gamma \sim \gamma(x) I_d$.
- The reconstruction depends on the boundary, and inaccuracies cause severe errors: not adapted for absolute measurements.
- Useful for anomaly detection, with difference measurements.
- Contrast vs. size requires very sensitive EIT system.

Introduction

With suitable internal data, the impedance problem can be solved with Calculus III.

- $\nabla u_1, \ldots, \nabla u_d$ are $C^2(\Omega)$,
- $\det(\nabla u_1, \ldots, \nabla u_d) \neq 0$ everywhere in Ω,

there is at most one $\gamma \in C^1(\Omega)$ (or less) such that for all $1 \leq i \leq d$,

$$\operatorname{div}(\gamma \nabla u_i) = 0 \text{ in } \Omega,$$

and $\int_\Omega a(x)dx = 1$.

Proof

Suppose $\gamma \in C^1$:

- $\mathrm{div}\left(\gamma \nabla u_{1,\dots,d}\right) = 0 \Leftrightarrow \mathrm{div}\left(\nabla u_{1,\dots,d}\right) + \nabla \log(\gamma) \cdot \nabla u_{1,\dots,d} = 0$
- $\left(\nabla \log(\gamma)\right)^T \left[\nabla u_1, \dots, \nabla u_d\right] = -\left[\mathrm{div}\left(\nabla u_1\right), \dots, \mathrm{div}\left(\nabla u_d\right)\right]$
- $\nabla \log(\gamma) = -\left(\left[\nabla u_1, \dots, \nabla u_d\right]^{-1}\right)^T \begin{bmatrix} \mathrm{div}\left(\nabla u_1\right) \\ \vdots \\ \mathrm{div}\left(\nabla u_d\right) \end{bmatrix}$

So γ is known up to a constant, fixed if $\int_\Omega a(x)dx = 1$... with an explicit pointwise formula.

Briane, Milton, Steibergs '13

Uniqueness is elusive.
Suppose $\nabla u_1, \ldots, \nabla u_d$ in $C^1(\mathbb{R}^d)$, and

$$\det\left[\nabla u_1, \ldots, \nabla u_d\right] > 0$$

everywhere.
Is there γ such that

$$\operatorname{div}\left(\gamma \nabla u_i\right) = 0 \text{ for all } i = 1, \ldots, d?$$

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- One can always find an *isotropic* periodic conductivity matching one field
- One can always find an *anisotropic* conductivity matching all fields.

EZ Caldéron

How to go from Calderón to the easy problem?
Questions to be solved

(a) Obtain $\nabla v_1, \ldots, \nabla v_d$ *(or other local informations)* from boundary *(or far field)* measurements.

(b) Ensure *a priori* that $\det(\nabla v_1, \nabla v_2, \ldots, \nabla v_d) \neq 0$ everywhere in (a subset of) Ω *(or other local contraints)*.

(c) Check consistency (regularity..)

(a) is Hybrid Imaging
(c) is regularity theory, modelling..
(b) is the focus of this talk.

Jacobian Constraints

(b) Ensure *a priori* that $\det(\nabla v_1, \nabla v_2, \ldots, \nabla v_d) \neq 0$ everywhere in (a subset of) Ω.

When $d = 2$, (b) is easy thanks to Alessandrini & Nesi '01 (quasi-conformal maps). In particular, if

$$\Delta u_i = 0 \text{ in } \Omega, \quad u_i = \phi_i \text{ on } \partial\Omega$$

satisfies $\det(\nabla u_1, \nabla u_2) > 0$ in Ω, then for any $\gamma \in L^\infty(\Omega)$, $\alpha < \gamma(x) < \beta$,

$$\mathrm{div}(\gamma \nabla v_i) = 0 \text{ in } \Omega, \quad v_i = \phi_i \text{ on } \partial\Omega$$

satisfy $\log\det(\nabla v_1, \nabla v_2) \in BMO(D)$ for every $D \Subset \Omega$.

Jacobian Constraints

When $d = 3$? Counterexamples exists to all 'homeomorphism' results (Wood '74 '91, Melas '82 Laugesen '96) even for the Laplacian.

- Anecdotal pathological examples ?
- Not uncommon situations, but which can be addressed, possibly of limited practical importance ?
- Illustration of an unavoidable, essential, roadblock ?

Jacobian Constraints

CGO Solutions (Calderón, Sylvester-Uhlmann '87, Bal-Uhlmann '09, Bal-Bonnetier-Monard-Triki '13):
Assume $\gamma \in H^{\frac{9}{2}+\varepsilon}(\Omega)$.
There exists a non-empty open set $\mathbf{G} \subset (H^{\frac{1}{2}}(\partial\Omega))^4$ of quadruples of illuminations such that for any $G = (g_1, g_2, g_3, g_4) \in \mathbf{G}$, there exists an open cover of Ω of the form $\{\Omega_{2i-1}, \Omega_{2i}\}_{1 \leq i \leq N}$ and a constant $c_0 > 0$ such that

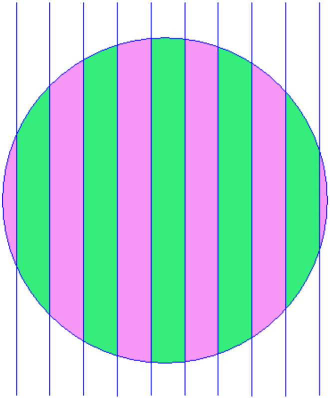
$$\inf_{x \in \Omega_{2i-1}} \det(\nabla u_1, \nabla u_2, \tilde{\epsilon}_i \nabla u_4) \geq c_0$$

and

$$\inf_{x \in \Omega_{2i}} \det(\nabla u_1, \nabla u_2, \epsilon_i \nabla u_3) \geq c_0, \quad 1 \leq i \leq N.$$

for ϵ_i and $\tilde{\epsilon}_i$ equal to ± 1.

Jacobian Constraints



If $\frac{a}{b} = \gamma$ when $b \neq 0$ and $\frac{c}{d} = \gamma$ when $d \neq 0$

and $b + d \neq 0$ then $\frac{a+c}{b+d} = \gamma$.

Jacobian Constraints

For $\phi = (\phi_1, \phi_2, \phi_3) \in H^{\frac{1}{2}}(\partial\Omega)$, let

$$\Delta U_0 = 0 \text{ in } \Omega, \quad U_0 = \phi \text{ on } \partial\Omega$$

and

$$\operatorname{div}\left(\gamma(n^3 x)\nabla U_n\right) = 0 \text{ in } \Omega \quad U_n = \phi \text{ on } \partial\Omega$$

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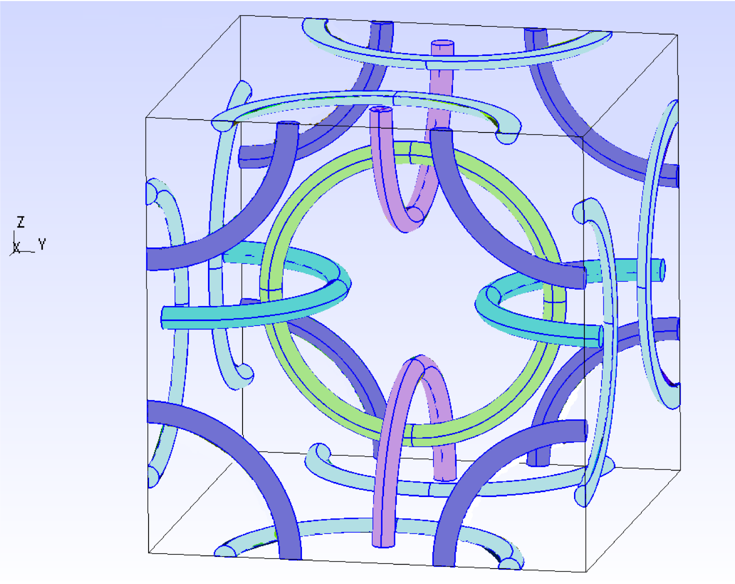
Theorem (C.). Let
$A := \left\{\phi \in H^{\frac{1}{2}}\left(\partial\Omega; \mathbb{R}^3\right) : \det\left(\nabla U_0\right) > \lambda \left\|\phi\right\|^3_{H^{\frac{1}{2}}(\partial\Omega)} \text{ in } B_\rho(x_0)\right\}$,
$\exists n\left(\rho, \Omega, \Omega', \lambda\right)$, $\tau > 0$, and

$$B_\pm \subset B(x_0, \rho) \text{ with } |B_\pm| > \tau |B(x_0, \rho)|$$

such that

$$\forall \phi \in A, \pm\det\left(\nabla U_n\right)(x) > \tau\lambda \left\|\phi\right\|^3_{H^{\frac{1}{2}}(\partial\Omega)} \text{ on } B_\pm,$$

Chainmail, Briane-Milton-Nesi ’04,’09



Proof

Briane & Milton & Nesi '04, ' 09: We can choose the constant γ_1 inside the links so that there exists Y_+ and Y_- open subsets of Y both of measure 2τ such that $P = \nabla\zeta$, where ζ is the solution of

$$\begin{array}{rcl} \mathrm{div}\,(\gamma\nabla\zeta) & = & 0 \text{ in } \mathbb{R}^3, \\ \zeta(y) - y & & H^1_\#(Y), \end{array}$$

satisfies

$$\det(P)(y) > 2\tau \text{ in } Y_+ \text{ and } \det(P)(y) < -2\tau \text{ in } Y_-.$$

Proof

Li & Nirenberg '03, Ben Hassen & Bonnetier '06: There exists a constant $C > 0$, independent of n such that

$$\begin{aligned} \|\nabla U_n\|_{L^\infty(\Omega')} &\leq C\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}, \\ \|P\left(n^3 x\right)\|_{L^\infty(\Omega')} &\leq C, \end{aligned}$$

and

$$\|\nabla U_n - P\left(n^3 x\right)\nabla U_0\|_{L^\infty(\Omega')} \leq \frac{1}{n} C\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}.$$

Proof

$$\begin{aligned}\det\left(\nabla U_n\right) &= \det\left(P\left(n^3x\right)\nabla U_0\right) + R_n,\\ &= \det\left(P\left(n^3x\right)\right)\det\left(\nabla U_0\right) + R_n\end{aligned}$$

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Li-Nirenberg gives

$$R_n \leq \frac{C}{n}\|\phi\|^3_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}.$$

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In the ball $B(x_0,\rho)$, for n large enough there are many copies of $Y_\pm$: call these sets $B_\pm$. $|B_\pm| > \tau\,|B(x_0,\rho)|$.

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Altogether in $B_\pm$

$$\begin{aligned}\pm\det\left(\nabla U_n\right) &> \left(2\tau\lambda - \frac{C}{n}\right)\|\phi\|^3_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}\\ &\geq \tau\lambda\|\phi\|^3_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}.\end{aligned}$$

(b) Altogether, Jacobian Constraints for the conductivity problem are hopeless..

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Back to (a), Hybrid Imaging Methods based on $2+$ **wave interactions**.

What if the wave(s) frequencie(s) are allowed to vary ?

$\rightsquigarrow$ Revisit (b) using multiple frequencies.

Not limited to Jacobians, or the conductivity equation. Maxwell, Elasticity, other non zero constraints.

Model Problem: microwave hybrid

Toy model : Helmholtz equation

$$\begin{cases} \nabla \cdot (\gamma \nabla u) + k^2 q u & = 0 \text{ dans } \Omega, \\ \gamma \frac{\partial u}{\partial \nu} & = g \text{ sur } \partial\Omega. \end{cases}$$

Suppose we know for a few $k \in (k_0, k_1)$ and φ

$$E[k, \varphi](x) := \gamma(x) \left|\nabla u(z)\right|^2 \text{ and } e[k, \varphi](z) := q(z)|u(z)|^2.$$

Find γ and q.

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- No global positivity on the Jacobian for a given frequency.
- 'If' there was one, algebraic solution (or other stable approaches).
- Can we combine various frequency data to guarantee that some satisfy it?

Model Problem: microwave hybrid

A proper set of measurement is such that

(i) For all $z \in D$,

$$1 \leq \sum_{i=1}^{N} e\left[k_i, \varphi_i\right](z) \leq M; \qquad 1 \leq \sum_{i=1}^{N} E\left[k_i, \varphi_i\right](z) \leq M;$$

(ii) For each $z \in D$ there exists i, j, l (changing with z) such that

$$\left|\det \begin{pmatrix} \nabla u\left[k_i, \varphi_i\right](z) & \nabla u\left[k_i, \varphi_j\right](z) & \nabla u\left[k_i, \varphi_l\right](z) \\ u\left[k_i, \varphi_i\right](z) & u\left[k_i, \varphi_j\right](z) & u\left[k_i, \varphi_l\right](z) \end{pmatrix}\right| \geq m^{3/2}.$$

(Ammari-deGournay-C-Rozanova-Triki '11)

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Do such sets exists, for given a priori boundary data?

Quantitative result of Giovanni Alberti

Given (k_0, k_1) an interval of possible frequencies. Given N, set

$$k_n = k_0 + \frac{n}{N+1}(k_1 - k_0) \quad n = 1, \ldots, N$$

and corresponding solutions u_i^n, $i = 0, \ldots, d$, $n = 1, \ldots N$,

$$\Delta u_i^n + k^n q u_i = 0 \text{ in } \Omega \quad u_i = x_i \text{ on } \partial\Omega \quad (x_0 = 1)$$

suppose $q \in L^\infty(\Omega)$, $0 < \alpha < q < \beta < \infty$.

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Théorème (G. S. Alberti '14). There exists $N(\alpha, \beta, \Omega, k_1, k_0)$ and $C(\alpha, \beta, \Omega, k_1, k_0)$ such that for each x there is $k \in \{k_1, \ldots, k_N\}$ such that $|u_0^k(x)| > C$, and

$$\left| \det \begin{bmatrix} u_1^k & \cdots & u_d^k \\ \nabla u_1^k & & \nabla u_d^k \end{bmatrix} \right| > C.$$

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Can be extended to $a \neq 1$.

Quantitative result of Giovanni Alberti : about the proof

- The constraint is satisfied at $k = 0$.
- So at no single point x can the constraint not be satisfied for sufficiently many k.
- Covering argument
- Can be made precise and quantitative (explicit bounds).

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But how many frequencies do you really need?

Proper set of measurement

Theorem (G.S. Alberti -C ' 14). Under the usual ellipticity assumptions, **suppose** γ, q **are analytic**. Given a range of possible frequencies $\mathcal{A}$,
$\Omega' \Subset \Omega$ and $f^1, \ldots, f^{d+1} \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$. If

$$F\left(0, f^1, \ldots, f^{d+1}\right)(x) \neq 0, \qquad x \in \Omega, \tag{1}$$

then

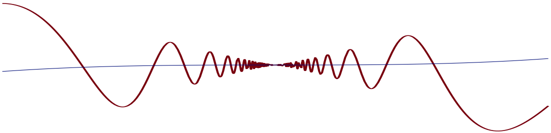
$$\left\{(\omega_1, \ldots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \{\omega_k\}_k \times \{f^1, \ldots, f^{d+1}\} \text{ is } F\text{-complete in } \Omega'\right\}$$

is open and dense in $\mathcal{A}^{d+1}$.

About the proof

Take for example $F(\omega, f^1, \ldots, f^{d+1}) = u[\omega, f^1](x)^2$,

- No accumulation points in $f(\omega_0, \ldots) = 0 \cap f(\omega_1, \ldots) = 0$: it is a finite set



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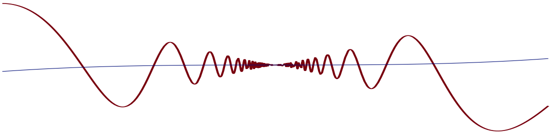

- Only a finite number of frequencies may cancel in that point.