Methods and Challenges in Extremal and Probabilistic Combinatorics*

Organizers

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1 Overview

Combinatorics, or discrete mathematics, is a fundamental mathematical discipline, concerned with the study of discrete mathematical objects such as graphs, set families and permutations, their typical and extremal properties, and their enumeration. A natural mathematical framework for a large variety of human activities and endeavors, combinatorics has been in existence for thousands of years. The field has experienced tremendous growth in the last few decades, having matured into an extremely important branch of mathematics with its own solid footing, theory, problems and set of tools, yet tightly and fruitfully interconnected with several other mathematical disciplines, as well as with theoretical computer science, where combinatorics in fact provides the ubiquitous mathematical basis. The maturity of combinatorics manifests itself in a multitude of ways, ranging from deep, powerful and well developed mathematical tools to extremely prolific interconnections between its subdisciplines. Leading researchers in the field do not perceive themselves as confined to one particular quarter of combinatorics, such as extremal graph theory, Ramsey theory or random graphs, but rather conduct wide ranging research programs addressing problems and actively utilizing methods from a wide variety of combinatorial disciplines. In fact, this unity and multi-faceted nature of discrete mathematics has served as the driving force behind the remarkable development of combinatorics in the last couple of decades, helping at the same time to attract bright young minds to the field.

Modern combinatorics is far too wide and diverse to have a workshop addressing all of its branches and touching upon all of its recent major developments. In this workshop therefore we confined ourselves mostly to extremal and probabilistic combinatorics, sometimes also known as Hungarian-type combinatorics to honor the enormous contribution of the Hungarian combinatorial school, led by Paul Erdős, to this field. These two disciplines (extremal combinatorics, probabilistic combinatorics) are tightly intertwined, sharing methodology, approaches and tools; it is quite standard to apply probabilistic tools to an extremal problem, or to study extremal properties of random structures, borrowing tools from extremal graph theory.

The activities of the workshop could be loosely classified into the following topics.

Extremal theory of graphs and hypergraphs is one of the central branches of discrete mathematics. It deals with the problem of determining or estimating the maximum or minimum possible number of edges in a graph or hypergraph satisfying certain restrictions. Extremal problems appear naturally in many areas such as discrete geometry, additive number theory, probability, analysis, computer science and coding theory.

Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is captured succinctly by the statement that "Every large system contains a large well organized subsystem". This is an

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area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to combinatorics but also to logic, analysis, number theory, and geometry. Since the publication of the seminal paper of Ramsey in 1930, this subject has experienced tremendous growth, and is currently among the most active areas in combinatorics.

One of the highlights of the workshop was the presentation by Choongbum Lee of a proof of the decades-old conjecture of Burr and Erdős, that very sparse graphs (of constant degeneracy) have linear Ramsey numbers. This longstanding open problem has motivated a great deal of important research over the years, and its solution is a very exciting development.

Probabilistic combinatorics studies probability spaces of discrete structures. Quite frequently these probability spaces are deep and complex enough to justify their study for their own sake, yet probabilistic considerations are of crucial value for many other areas of combinatorics, most notably for extremal problems and Ramsey theory. They also constitute a mathematical foundation for design and analysis of randomized algorithms, and more generally for addressing the fundamental role of randomness in theoretical computer science. Probabilistic methods have grown to be an essential tool in the arsenal of every combinatorialist, and their power, versatility and applicability have been tremendous.

In the remainder of this report we present in detail some of the advances presented at the workshop.

2 Extremal Theory

Augmented trees with high girth

Noga Alon

Let G be a graph consisting of a complete binary tree of depth h together with one back edge leading from each leaf to one of its ancestors, and suppose that the girth of G exceeds g. Let $h = h(g)$ be the minimum possible depth of the underlying binary tree in such a graph. The existence of such graphs, for arbitrarily large g, is proved in [1]. As shown there, these graphs provide simple explicit constructions of graphs and hypergraphs of high girth and high chromatic number, and are also useful in the investigation of several natural list coloring problems. The graphs constructed in that paper are huge, their number of vertices grows like some version of the Ackermann function. It is also proved in [1] that the number of vertices must be large, specifically, $h(g)$ is at least a tower of height $2^{\Omega(g)}$. Although that is a fast growing function of g, the upper bound is far, far larger. Here we show that the upper bound is closer to the correct behavior, and the value of $h(g)$ is indeed Ackermannian.

In view of of its rapid growth, it may be interesting to try and compute or estimate $h(g)$ more accurately for small values of g. It is not difficult to check that $h(1) = 1, h(2) = 2, h(3) = 3, h(4) = 5, h(5) = 7, h(6) = 10$ and $h(7) = 13$. However, despite this modest growth at the beginning, $h(g)$ is much larger for bigger values of g. In particular, without trying to optimize the computation very carefully, we can show that

$$h(16) > Tower(2^{2^{2^{18}}}).$$

Triangle factors of graphs without large independent sets and of weighted graphs

Jozsef Balogh *joint result with Molla and Sharifzadeh*

The classical Corrádi-Hajnal theorem [6] claims that every n-vertex graph G with $\delta(G) \geq 2n/3$ contains a triangle factor, when $3|n$. In this paper we asymptotically determine the minimum degree condition necessary to guarantee a triangle factor in graphs with sublinear independence number. In particular, we show that if G is an n-vertex graph with $\alpha(G) = o(n)$ and $\delta(G) \geq (1/2+o(1))n$, then G has a triangle factor and this is asymptotically best possible. Furthermore, it is shown a result related to the Hajnal-Szemerédi Theorem [12], that for every r

that if every linear size vertex set of a graph G spans quadratic many edges, and $\delta(G) \geq (1/2 + o(1))n$, then G has a K_r-factor for n sufficiently large. We also propose many related open problems whose solutions could show a relationship with Ramsey-Turán theory.

Additionally, we also consider a fractional variant of the Corrádi-Hajnal Theorem, settling a conjecture of Balogh-Kemkes-Lee-Young. Let $t \in (0, 1)$ and $w : E(K_n) \to [0, 1]$. We call a triangle in K_n heavy if the sum of the weights on its edges is more than $3t$. We prove that if $3|n$ and w is such that for every vertex v the sum of $w(e)$ over edges e incident to v is at least $\left(\frac{1+2t}{3} + o(1)\right) n$, then there are $n/3$ vertex disjoint heavy triangles in G. The methods are using the absorbing technique of Rödl, Ruciński, and Szemerédi [17].

Ranks of matrices with few distinct entries

Boris Bukh

There are many applications of linear algebra to combinatorics that follow the same recipe. They begin with n objects of some kind, and a desire to bound n. One then maps each of these objects to a pair $(v_i, u_i) \in V \times V^*$ where V and V^* are a vector space and its dual. The map is chosen so that the rank of the n-by-n matrix $M = (u_i v_j)_{i,j}$ is large whenever n is large. Since rank $M \leq \dim V$, that yields a bound on n. In many of these applications V is an inner product space, and $v_i = u_i$, but it is not always the case.

The applications of this recipe include the proofs of the non-uniform Fisher inequality, the Frankl–Wilson bound on L-intersecting families, Haemers' bound on the Shannon capacity of a graph and bounds on s-distance sets. More applications can be found in the books by Babai- Frankl [2] and by Matoušek [13].

In all the applications named above, the matrices which arise are of a special form — all diagonal entries are equal, and the off-diagonal entries take on boundedly many distinct values. It is this property that is used to bound their rank. The bounds on the ranks of such matrices are is subject of the present talk.

We define an *L-matrix* to be a square matrix whose diagonal entries are all equal to 0, and each of whose off-diagonal entries is an element of the set L.

$$N(r, L) = \max\{n : \exists\ n\text{-by-}n\ L\text{-matrix of rank } \leq r\}.$$

We characterize the sets L for which $N(r, L) \leq r + 1$, as well as the sets L for which $N(r, L) = O(r)$.

As a by-product of our results we prove, for each λ, the sharp asymptotic bounds on the maximum multiplicity of λ as an eigenvalue of an n-vertex digraph.

Rational exponents in extremal graph theory

David Conlon *joint work with Boris Bukh*

Given a family of graphs $\mathcal{H}$, the extremal number $\mathrm{ex}(n, \mathcal{H})$ is the largest m for which there exists a graph with n vertices and m edges containing no graph from the family $\mathcal{H}$ as a subgraph. We show that for every rational number r between 1 and 2, there is a family of graphs $\mathcal{H}_r$ such that $\mathrm{ex}(n, \mathcal{H}_r) = \Theta(n^r)$. This solves a longstanding open problem in extremal graph theory that has been reiterated by a number of authors, including Frankl [9] and Füredi and Simonovits [10]. The proof of this result uses a variant of the random algebraic technique introduced by Blagojević, Bukh and Karasev and by Bukh.

A removal lemma for nearly-intersecting families

Shagnik Das *joint work with Tuan Tran*

A k-uniform family of subsets of $[n]$ is *intersecting* if it does not contain a disjoint pair of sets. The study of intersecting families is central to extremal set theory, dating back to the seminal Erdős–Ko–Rado theorem of 1961 that bounds the largest such families. A recent trend has been to investigate the structure of set families with few disjoint pairs.

Friedgut and Regev proved a general removal lemma, showing that when $\gamma n \leq k \leq \left(\frac{1}{2} - \gamma\right) n$, a set family with few disjoint pairs can be made intersecting by removing few sets. We were able to provide a simple proof of a special case of this theorem, when the family has size close to $\binom{n-1}{k-1}$. However, our theorem holds for all $2 \leq k < \frac{1}{2}n$ and provides sharp quantitative estimates. Our proof uses spectral analysis and recent work of Filmus, who proved an analogue of the Friedgut–Kalai–Naor theorem for uniform slices of the hypercube.

As an application of our removal lemma, we settle a question of Bollobás, Narayanan and Raigorodskii regarding the independence number of random subgraphs of the Kneser graph $K(n,k)$. The Erdős–Ko–Rado theorem shows $\alpha(K(n,k)) = \binom{n-1}{k-1}$. For some constant $c > 0$ and $k \leq cn$, we determine the sharp threshold for when this equality holds for random subgraphs of $K(n,k)$, and provide strong bounds on the critical probability for $k \leq \frac{1}{2}(n-3)$. This extends results of Bollobás, Narayanan and Raigorodskii and also of Balogh, Bollobás and Narayanan.

Entropy as a tool for proving geometric and analytical inequalities

Ehud Friedgut *joint work with Ellis, Kindler, and Yehudayoff)*

The Loomis-Whitney inequality, which bounds the volume of an n-dimensional body in terms of the volumes of its $(n-1)$-dimensional projections, and the Bollobas-Thomason inequalities which generalize it, are known [folklore, probably due to Radhakrishnan] to follow from entropy considerations. We will see in this talk how to derive stability versions using this approach, together with Pinsker's information-theoretical inequality. In a nutshell: the inequalities are tight only if the body in question is a box, and we show that if the inequalities are close to being tight then the body must be close to a box.

We will also see how to use entropy considerations to derive the famous Bomami-Gross-Beckner inequality, one of the most useful inequalities in analysis of Boolean functions. In this approach the induction on the dimension is replaced by using the chain rule for entropy, and studying the information revealed by studying the bits of two correlated vectors as they are exposed sequentially.

On the boundary of the region defined by subgraph densities

Hamed Hatami

Many fundamental theorems in extremal graph theory can be expressed as relations between subgraph densities. For dense graphs, it is possible to replace subgraph densities with homomorphism densities. Inspired by the work of Freedman, Lovász and Schrijver, in recent years, a new line of research in the direction of treating and understanding these relations in a unified way has emerged. Despite all the these investigations there are very few general results known about such relations. An early result of Erdos, Lovasz, and Spencer shows that the region defined by these inequalities has non-empty interior. It is also easy to see that it is a connected region. It is not even known whether the region is simply connected.

For the special case of the region defined by edge and triangle densities, the Kruskal-Katona theorem and a recent theorem of Razborov show that the closure of the boundary of the set of points defined by the edge and the triangle homomorphism densities of finite graph is a countable set of algebraic curves [16]. In particular, it is almost everywhere smooth. This raises the very natural question that whether the boundary is always as well-behaved even if one considers other graphs instead of an edge and a triangle. We construct examples that show that the (restrictions) of the boundary can have nowhere differentiable parts.

Digraphs of large girth with every small subset dominated

Hao Huang *joint work with Yogesh Anbalagan, Shachar Lovett, Sergey Norin, Adrian Vetta and Hehui Wu*

A tournament T is a directed graph obtained by assigning a direction to each edge in a complete graph. A subset of vertices S is acyclic if the induced subgraph of T on S contains no directed cycle. The chromatic number $\chi(T)$ is the minimum size of a partition of $V(T)$ into acyclic sets. Denote by $N^+(v)$ the out-neighborhood of the vertex v. Berger et al. posed the following conjecture on tournament coloring: for each k there exists a constant C_k such that if T is a tournament in which $\chi(T[N^+(v)]) \leq k$ for each vertex v in T, then $\chi(T) \leq C_k$. Note that a similar statement does not hold for undirected graphs, since there exist triangle-free graphs with arbitrarily high chromatic number like the Mycielski graphs. The tournament coloring conjecture has only been solved for $k = 1$. One could imitate the idea of its proof and it turns out that the following stronger conjecture, posed earlier by Myers, and recently by Daskalakis, Mehta, and Papadimitriou, motivated from their research in game theory, would imply the tournament coloring conjecture.

Conjecture. *There exists integers k, l such that every directed graph either has a cycle of length at most k, or an undominated set of l vertices.*

In this talk, we present a counterexample to this conjecture, based on an addictive combinatorial construction by Haight. Our counterexample also leads to a negative answer to an outstanding open problem in game theory: is there a constant k and an $\varepsilon < 1$ such that, for any bimatrix game, there is a ε-well supported Nash equilibrium with supports of cardinality at most k. This result illustrates a fundamental structural distinction between ε-well supported Nash equilibrium and ε-Nash equilibrium. This structural distinction also has practical implications with regards to behavioural models and popular equilibria search algorithms that focus upon small supports.

On the Corradi-Hajnal Theorem and a question of Dirac

Alexandr Kostochka *joint work with Hal Kierstead, Theodore Molla, and Elyse Yeager*

In 1963, Corrádi and Hajnal proved Erdős' conjecture that for all $k \geq 1$ and $n \geq 3k$, every (simple) graph G on n vertices with minimum degree $\delta(G) \geq 2k$ contains k (vertex-)disjoint cycles. This sharp result inspired a series of generalizations and refinements. In particular, the same year, Dirac described the 3-connected multigraphs not containing two disjoint cycles and asked the more general question: Which $(2k-1)$-connected multigraphs do not contain k disjoint cycles?

Enomoto and independently Wang proved the following Ore-type refinement: For all $k \geq 1$ and $n \geq 3k$, every (simple) graph G on n vertices such that $d(x) + d(y) \geq 4k - 1$ for all non-adjacent $x, y \in V(G)$ contains k disjoint cycles.

The goal of the talk is twofold. First, we describe all simple n-vertex graphs G with $d(x) + d(y) \geq 4k - 3$ for all non-adjacent $x, y \in V(G)$ that do not have k disjoint cycles. Then we use this result to answer Dirac's question in full: we describe the multigraphs that do not have k disjoint cycles in a wider class, namely, in the class of multigraphs in which every vertex has at least $2k - 1$ distinct neighbors.

Edge-decompositions of graphs with high minimum degree

Deryk Osthus *joint work with Ben Barber, Daniela Kühn, and Allan Lo*

A fundamental theorem of Wilson states that, for every graph F, every sufficiently large F-divisible clique has an F-decomposition. Here a graph G is F-divisible if $e(F)$ divides $e(G)$ and the greatest common divisor of the degrees of F divides the greatest common divisor of the degrees of G, and G has an F-decomposition if the edges of G can be covered by edge-disjoint copies of F.

We extend this result to graphs G which are allowed to be far from complete. In particular, together with a result of Dross on fractional triangle decompositions, our results imply that every sufficiently large K_3-divisible graph of minimum degree at least $9n/10 + o(n)$ has a K_3-decomposition. This significantly improves previous results towards the long-standing conjecture of Nash-Williams that every sufficiently large K_3-divisible graph with minimum degree at least $3n/4$ has a K_3-decomposition. We also obtain the asymptotically correct minimum degree thresholds of $2n/3 + o(n)$ for the existence of a C_4 decomposition, and of $n/2 + o(n)$ for the existence of a $C_{2\ell}$-decomposition, where $\ell \geq 3$.

Our main contribution is a general 'iterative absorption' method which turns an approximate or fractional decomposition into an exact one. In particular, our results imply that in order to prove an asymptotic version of Nash-Williams' conjecture, it suffices to show that every K_3-divisible graph with minimum degree at least $3n/4 + o(n)$ has an approximate K_3-decomposition. More generally, we can combine our results with recent ones on fractional decompositions of cliques obtained in [3].

Supersaturation Problem for Colour-Critical Graphs

Oleg Pikhurko *joint work with Zelealem B. Yilma*

The *Turán function* $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an F-free graph with n vertices. The classical results of Turán and Rademacher from 1941 led to the study of supersaturated graphs where the key question is to determine $h_F(n, q)$, the minimum number of copies of F that a graph with n vertices and $\mathrm{ex}(n, F) + q$ edges can have.

Here, let F be an *r-colour-critical* graph (that is, F contains an edge whose deletion reduces its chromatic number from $r + 1$ to r). Simonovits proved that, for all large n, the *Turán graph* $T_r(n)$, the complete almost-balanced r-partite graph of order n, is the unique extremal graph for $\mathrm{ex}(n, F)$.

Determining the value of $h_F(n, q)$ is difficult in general even for small q. For example, let c_1 be the limit superior of q/n for which the extremal structures are obtained by adding some q edges to $T_r(n)$. The problem of determining c_1 for cliques was a well-known question of Erdős that was solved only decades later by Lovász and Simonovits. Extending results of Mubayi [14], we prove that $c_1 > 0$ for every r-colour-critical F. Our approach also allows us to determine c_1 for a number of graphs, including odd cycles, cliques with one edge removed, and complete bipartite graphs plus an edge. Unfortunately, there seems to be no nice general theory that gives the value of c_1 as this parameter is usually determined by lower-order terms and we have examples of small explicit graphs F for which c_1 shows a complicated behaviour.

On the other hand, the asymptotic behaviour of $h_F(n, q)$ (where one is interested in its value within factor $1 + o(1)$) is easier to analyse: we solve this problem for every r-colour-critical F when $q = o(n^2)$. In particular, we can determine the liminf as $n \to \infty$ of q_0/n, where $q_0 = q_0(n, F)$ is the smallest q such that $T_r(n)$ plus any q extra edges has strictly more than $(1 + o(1))h_F(n, q)$ copies of F.

Graphs of large chromatic number

Alex Scott *joint work with Paul Seymour and Maria Chudnovsky*

A *hole* in a graph G is an induced subgraph which is a cycle of length at least four, and an *odd hole* means a hole of odd length. In 1985, A. Gyárfás made three famous conjectures:

Conjecture. *For every integer $k > 0$, every graph of sufficiently large chromatic number contains either a complete subgraph on k vertices or an odd hole.*

Conjecture. *For all integers $k, t > 0$, every graph of sufficiently large chromatic number contains either a complete subgraph on k vertices or a hole of length at least t.*

Conjecture. *For all integers $k, t > 0$, every graph of sufficiently large chromatic number contains either a complete subgraph on k vertices or an odd hole of length at least t.*

Scott and Seymour recently proved the first conjecture, and also the third conjecture when $k = 3$. (In fact we proved much more: for all $t \geq 0$, in every graph with large enough chromatic number and no triangle, there are holes of t consecutive lengths.). The third conjecture implies the other two, and remains open. In this talk we present a proof of the second conjecture.

Decomposing a Graph Into Expanding Subgraphs

Asaf Shapira *joint work with Guy Moshkovitz*

A paradigm that was successfully applied in the study of both pure and algorithmic problems in graph theory can be colloquially summarized as stating that *any graph is close to being the disjoint union of expanders.* We prove that in several of the instantiations of the above approach, the quantitative bounds that were obtained are essentially best possible. Our results include:

- A classical result of Lipton, Rose and Tarjan from 1979 states that if $\mathcal{F}$ is a hereditary family of graphs and every graph in $\mathcal{F}$ has a vertex separator of size $n/(\log n)^{1+o(1)}$, then every graph in $\mathcal{F}$ has $O(n)$ edges.

We construct a hereditary family of graphs with vertex separators of size $n/(\log n)^{1-o(1)}$ such that not all graphs in the family have $O(n)$ edges.

- Sudakov and Shapira have recently shown that every graph of average degree d contains an n-vertex graph of average degree $(1-o(1))d$ and vertex expansion $1/\log^{1+o(1)} n$. We show that one cannot get better vertex expansion, even if allowing average degree $o(d)$.

- Motivated by the Unique Games Conjecture, Trevisan and Arora, Barak and Steurer showed that given a graph G, one can remove only 1% of G's edges and thus obtain a graph in which each connected component has good expansion properties. We show that in both of these decomposition results, the expansion properties they guarantee are (essentially) best possible, even when one is allowed to remove 99% of G's edges. In particular, our results imply that the eigenspace enumeration approach of Arora-Barak-Steurer cannot give (even quasi-) polynomial time algorithms for unique games.

All our results are obtained as corollaries of a new family of graphs, constructed by picking random subgraphs of the hypercube, and analyzed using arguments from the theory of metric embedding.

3 Probabilistic Combinatorics

Online sprinkling and packing problems
Asaf Ferber

In this talk we introduced the notion of *online sprinkling*, due to Van Vu and myself. Roughly speaking, in an online sprinkling our goal is to create a randomized algorithm that finds a large structure in $H \sim \mathcal{H}^k(n,p)$ (where $\mathcal{H}^k(n,p)$ is the binomial k-uniform hypergraph). We aim to find the target structure as a subgraph of the "online generated" random hypergraph H. That is, during the execution of the algorithm, a random hypergraph is being generated and the target structure is being constructed together step by step. The way the algorithm works is as follows: in each time step i of the algorithm, a subset $E_i \subseteq E(k_n)$ is being chosen according to some distribution, and next we choose every edge in E_i (independently) with some probability p_i. All the chosen edges will be part of the random hypergraph.

For each k-tuple $e \in \binom{[n]}{k}$, let

$$\omega(e) = 1 - \prod_{i: e \in E_i} (1 - p_i)$$

be the *weight* of e at the end of the algorithm. Clearly, if $\omega(e) \leq p$ for each k-tuple e, then the resulting hypergraph can be coupled as a subgraph of $H \sim \mathcal{H}^k(n,p)$.

Although the approach sounds natural and relatively simple, it proved itself to be quite powerful. We briefly described the relatively elegant proofs of the following two results:

(a) Packing perfect matchings in $\mathcal{H}^k(n,p)$ in nearly optimal p (joint with Van Vu). This may be seen as a generalization of a result of Johannson, Kahn and Vu.

(b) Packing spanning trees of max degree $(np)^{1/10}$ in $\mathcal{G}(n,p)$ for $p \geq \log^C n/n$ (joint with Choongbum Lee and Wojtek Samotij). This result is currently the strongest known result towards the so called Gyarfas-Lehel Conjecture [11].

Few other interesting and easy applications have been mentioned (packing loose Hamilton cycles, packing Hamilton cycles oriented arbitrarily, etc.) and we guess that this method will start being more popular in the following couple of years.

Optimal path and cycle decompositions of dense quasirandom graphs

Daniela Kuhn *joint work with Stefan Glock and Deryk Osthus*

There are several longstanding and beautiful conjectures on decompositions of graphs into cycles and/or paths. We consider four of the most well-known in the setting of dense quasirandom and random graphs: the Erdős-Gallai conjecture on decompositions into cycles and edges, the Gallai conjecture on decompositions into paths, the linear arboricity conjecture on decompositions into linear forests as well as the overfull subgraph conjecture on edge-colourings. In particular, we prove the following optimal decomposition results for random graphs. Let $0 < p < 1$ be constant and let $G \sim G_{n,p}$. Let $odd(G)$ be the number of odd degree vertices in G. Then asymptotically almost surely the following hold:

(i) G can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles and a matching of size $odd(G)/2$.

(ii) G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.

(iii) G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.

Each of these bounds is best possible. We actually derive (i)–(iii) from 'quasirandom' versions of our results. There is also an interesting connection to the overfull subgraph conjecture on edge-colourings of graphs: we determine the edge chromatic number of a given dense quasirandom graph of even order, proving the overfull subgraph conjecture for such graphs. For all these results, our main tool is a result on Hamilton decompositions of robust expanders.

Random simplicial complexes - Progress report

Nathan Linial

The systematic study of random graphs was started by Erdős and Rényi in the early 1960's. Since a graph can be viewed as a one-dimensional simplicial complex, it is natural to seek an analogous theory of d-dimensional random simplicial complexes for all $d \geq 1$. Such an analog of Erdős and Rényi's $G(n, p)$ model, called $X_d(n, p)$, was introduced by Linial and Meshulam some 10 years ago. A simplicial complex X in this probability space is d-dimensional, it has n vertices and a full $(d-1)$-dimensional skeleton. Each d-face is placed in X independently with probability p. Note that $X_1(n, p)$ is identical with $G(n, p)$.

A main theme in $G(n, p)$ theory is the search for *threshold* probabilities. One of Erdős and Rényi's main discoveries is that $p = \frac{\ln n}{n}$ is the threshold for graph connectivity. Graph connectivity can be equivalently described as the vanishing of the zeroth homology, and this suggests a d-dimensional counterpart. Indeed, it was shown by Linial-Meshulam with subsequent work by Meshulam-Wallach that in $X_d(n, p)$ the threshold for the vanishing of the $(d-1)$-th homology is $p = \frac{d \ln n}{n}$.

But perhaps the most exciting early discovery in $G(n, p)$ theory is the so-called *phase transition* that occurs at $p = \frac{1}{n}$. This is where the random graph asymptotically almost-surely (a.a.s.) acquires cycles. Namely for $p = o(\frac{1}{n})$ a $G(n, p)$ graph is a.a.s. a forest. For every $1 > c > 0$ there is a $1 > q = q(c) > 0$ such that a graph in $G(n, \frac{c}{n})$ is a forest with probability $q + o_n(1)$. Finally, for $p \geq \frac{1}{n}$, a $G(n, p)$ graph has, a.a.s., at least one cycle. Moreover, at around $p = \frac{1}{n}$ the random $G(n, p)$ graph acquires a *giant component*, a connected component with $\Omega(n)$ vertices. The present work is motivated by the quest of d-dimensional analogs of these phenomena.

As is often the case when we consider the one vs. high-dimensional situations, the plot thickens here. Whereas acyclicity and collapsibility are equivalent for graphs, this is no longer the case for $d \geq 2$. Clearly, a d-collapsible simplicial complex has a trivial d-th homology, but the reverse implication does not hold in dimension $d \geq 2$. In this view, there are now two potentially separate thresholds to determine in $X_d(n, p)$: For d-collapsibility and for the vanishing of the d-th homology. Also, there is a conceptual difficulty here, since there is no natural notion of a connected component in dimensions 2 and above. These questions were answered in several papers written jointly with L. Aroshtam, T. Luczak, R. Meshulam and Y. Peled, see below. In my lecture I surveyed some of the main highlights of this research project.

Forcing quasirandomness with triangles

Mathias Schacht *joint work with Christian Reiher*

Thomason [18, 19] and Chung, Graham, and Wilson [5] initiated the systematic study of dense *quasirandom* graphs. Roughly speaking, a graph $G = (V, E)$ on $n = |V|$ vertices and with edge density $p = |E|/\binom{n}{2}$ is quasirandom, if it displays several properties that hold for the random graph $G(n, p)$ with probability tending to 1 as $n \to \infty$. For example, a key feature of the random graph is its uniform edge distribution, which can be rendered by the following definition: we say a sequence of graphs $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ with $|V_n| \to \infty$ is *p-quasirandom* if for every X, $Y \subseteq V_n$ we have

$$e(X, Y) = p\,|X|\,|Y| + o(|V_n|^2)\,, \tag{1}$$

where $e(X, Y)$ denotes the number of ordered pairs $(x, y) \in X \times Y$ such that $\{x, y\}$ is an edge of G_n. A substantial part of the study of quasirandom graphs concerns graph properties that are equivalent to being p-quasirandom. In the original work of Chung, Graham, and Wilson six such properties (besides the definition from above) were discussed and since then quite a few equivalent properties were found. In particular, already in [5] it was noted that a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is p-quasirandom if the graphs of the sequence have edge density at least $p + o(1)$ and their C_4-density (i.e., the number of cycles of length four in G_n divided by the number of four-cycles in the complete graph on the same number of vertices) is at most $p^4 + o(1)$. In other words, one may say that the *right* global number of C_4's in a graph G of given edge density *forces* it to be quasirandom (as it is common in this area we suppress the sequence of graphs here and simply imagine a sufficiently large graph G). It is well known that the graph C_4 cannot be replaced by an arbitrary graph F and it is an intriguing open question to characterise those graphs F that force quasirandomness just like the C_4 in the discussion above.

However, it was shown by Simonovits and Sós that quasirandomness is forced by any graph F with at least one edge, if we insist on the right F-density in a *hereditary* sense, i.e., if every subset of vertices induces a subgraph with the right F-density. Focusing just on the case when F is a triangle the result of Simonovits and Sós asserts that a graph $G = (V, E)$ is p-quasirandom if and only if for all subsets X, Y, $Z \subseteq V$ the number of triples $(x, y, z) \in X \times Y \times Z$ with the property that $\{x, y, z\}$ spans a triangle in G is $p^3\,|X|\,|Y|\,|Z| + o(|V|^3)$. Their original proof is based on Szemerédi's regularity lemma and we obtained a simple proof which avoids the use of the regularity lemma and yields a better numerical dependency of the involved error parameters. Another regularity-lemma-free proof of this result was independently achieved by Conlon, Fox, and Sudakov.

Furthermore, we obtained (using the regularity lemma again) a stronger result, which asserts that, in fact, it suffices to assume the correct triangle density induced by two subsets.

Theorem. *For every $p > 0$ and $\varepsilon > 0$ there exist $\delta > 0$ such that for every graph $G = (V, E)$ the following holds. If for all subsets X, $Y \subseteq V$ the number $K_3(X, Y, V)$ of triples $(x, y, v) \in X \times Y \times V$ such that $\{x, y, v\}$ spans a triangle in G satisfies*

$$\big|K_3(X, Y, V) - p^3\,|X|\,|Y|\,|V|\big| \le \delta|V|^3\,,$$

then G is p-quasirandom, i.e., (1) holds where $o(|V_n|^2)$ is replaced by $\varepsilon|V|^2$.

The proof of this result is somewhat specific for the triangle case and finding an adequate extension for arbitrary fixed graphs F appears to be an interesting open problem.

Sharp thresholds for half-random games

Tibor Szabó *joint work with Jonas Groschwitz*

Randomized strategies in deterministic positional games were studied sporadically in various setups, e.g. the H-building game of Bednarska and Łuczak or the giant component building game of the Achlioptas process. Here we initiate their systematic study for Maker-Breaker games. (Recently, Krivelevich and Kronenberg also studied the problem independently.) In a Maker-Breaker positional game over a finite hypergraph $\mathcal{F} \subseteq 2^X$ two players, Maker and Breaker, alternately take turns in occupying free elements of the vertex set X, until no elements of X is free. Maker is the winner if he completely occupies an member of $\mathcal{F}$, otherwise Breaker wins. We study biased Maker-Breaker positional games between two players, one of whom is is taking b uniformly random vertices in each round against an opponent with an optimal strategy, but taking only one vertex in each round. In both such scenarios, that is when Maker plays randomly and when Breaker plays randomly, we determine the sharp threshold bias of classical graph games, such as connectivity, Hamiltonicity, and minimum degree-k.

The traditional, deterministic version of these games, with two optimal players playing, are known to obey the so-called probabilistic intuition. That is, the threshold bias $b_{\mathcal{F}}$ (the smallest bias Breaker has a winning strategy with) for these games is asymptotically equal to the threshold bias of their random counterpart, where players just take edges uniformly at random [8]. We find, that despite this remarkably precise agreement of the results of the deterministic and the random games, playing randomly against an optimal opponent is not a good idea: the threshold bias becomes significantly higher in favor of the "clever" player. In the `CleverMaker-RandomBreaker` setup the sharp threshold bias turns out to be linear in n, as opposed to the $\frac{n}{\ln n}$ threshold bias of the fully random and fully clever variants. In the `RandomMaker-CleverBreaker` setup the sharp threshold bias is proved to be $\ln \ln n$ — but tilted in favor of `RandomMaker`. Thus the quantitative aspects of these games change quite a bit compared to the clever and the random variants. An important qualitative aspect of the probabilistic intuition carries through nevertheless: the bottleneck for Maker to occupy a connected graph is still the ability to avoid isolated vertices in her graph.

Jacques Verstraete *Full subgraphs*

Let $G = (V, E)$ be a graph of density p on n vertices. Following Erdős, Łuczak and Spencer [7], an m-vertex subgraph H of G is called *full* if H has minimum degree at least $p(m-1)$. Let $f(G)$ denote the order of a largest full subgraph of G. If $p\binom{n}{2}$ is a positive integer, define

$$f_p(n) = \min\{f(G) : |V(G)| = n,\ |E(G)| = p\tbinom{n}{2}\}.$$

Erdős, Łuczak and Spencer [7] proved that if $p = \frac{1}{2}$, then for $n \geq 2$,

$$\sqrt{2n} - 2 \leq f_p(n) \leq 4n^{\frac{2}{3}}(\log n)^{\frac{1}{3}}.$$

Here we show that for infinitely many p near the elements of $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$, $f_p(n) = \Theta(n^{\frac{2}{3}})$, and more generally that for all $p :\ n^{-\frac{2}{3}} < p < 1 - n^{-\frac{1}{5}}$ we have

$$f_p(n) = \Omega((1-p)^{\frac{1}{3}} n^{\frac{2}{3}}).$$

As an ingredient of the proof, we show that every graph G on n vertices has a subgraph H on m vertices, where $\lfloor \frac{n}{r} \rfloor \leq m \leq \lceil \frac{n}{r} \rceil + 1$, such that for every $v \in V(H)$, the degree of v in H is at least $1/r$ times its degree in G. A main open problem is to determine the order of magnitude of $f_p(n)$ when $p \notin \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$.We show that for all $p \geq n^{-\frac{2}{3}}$,

$$0.1n^{\frac{2}{3}} \leq f_p(n) \leq 10p^{\frac{1}{5}} n^{\frac{4}{5}} (\log n)^{\frac{1}{5}}.$$

Anti-concentration inequalities for polynomials

Van Vu *joint work with R. Meka and O. Nguyen*

An anti-concentration inequality bounds the probability that a random variable F takes value in a fixed small interval. The first such result is the famous Littlewood-Offord inequality, strengthened by Erdős. Let

ξ_i be iid Rademacher random variables and c_i be real coefficients with absolute value at least 1. Consider the linear function

$$F = c_1\xi_1 + \ldots + c_n\xi_n.$$

The ELO inequality asserts that for any fixed unit interval I, the probability that F belongs to I is $O(n^{-1/2})$. This is a beginning point of an extensive studies through several decades [15].

In this talk, we present a generalization of the ELO result to the case when F is a polynomial in terms of the variables ξ_i. Our bound is near optimal and improves significantly earlier estimates by Costello-Tao-Vu and Razborov-Viola. As applications, we settle (up to an iterative logarithmic term) a problem of Razborov and Viola in complexity theory and prove a result concerning the number of copies of a small fixed graph H in a random graph.

4 Ramsey Theory

Ramsey numbers of degenerate graphs

Choongbum Lee

For a graph H, the *Ramsey number* of H, denoted $r(H)$, is defined as the minimum integer n such that in every edge two-coloring of K_n, the complete graph on n vertices, there exists a monochromatic copy of H. There are many fascinating problems studying bounds on Ramsey numbers of various graphs. In 1973, Burr and Erdős [4] initiated the study of Ramsey numbers of sparse graphs and conjectured that the behavior of Ramsey numbers of sparse graphs should be dramatically different from that of complete graphs. A graph G is *d-degenerate* if all its subgraphs contain a vertex of degree at most d. Degeneracy is a natural measure of sparseness of graphs as it implies that for all subsets of vertices X, there are fewer than $d|X|$ edges with both endpoints in X. Burr and Erdős conjectured that for every natural number d, there exists a constant $c = c(d)$ such that every d-degenerate graph H on n vertices satisfies $r(H) \leq cn$. This is in striking contrast with the case of complete graphs where the dependence on number of vertices is exponential. This conjecture has received much attention and motivated several important developments over the past 40 years.

In this talk, we build upon previous developments and settle the conjecture of Burr and Erdős. We say that a graph G is *universal* for a family $\mathcal{F}$ of graphs if it contains all graphs $F \in \mathcal{F}$ as subgraphs. For an edge coloring of a graph, we say that a color is *universal* for a family $\mathcal{F}$ if the subgraph induced by the edges of that color is universal for $\mathcal{F}$.

Theorem. *There exists a constant c such that the following holds for every natural number d and r. For every edge two-coloring of the complete graph on at least $2^{d2^{cr}}n$ vertices, one of the colors is universal for the family of d-degenerate r-colorable graphs on at most n vertices.*

This settles the conjecture of Burr and Erdős since all d-degenerate graphs have chromatic number at most $d+1$. Moreover, for fixed values of r, the theorem is best possible up to the constant in the exponent.

Directed paths: from Ramsey to Ruzsa and Szemerédi

Po-Shen Loh

Starting from an innocent Ramsey-theoretic question regarding directed paths in tournaments, we discover a series of rich and surprising connections that lead into the theory around a fundamental problem in Combinatorics: the Ruzsa-Szemerédi induced matching problem. Using these relationships, we prove that every coloring of the edges of the transitive n-vertex tournament using three colors contains a directed path of length at least $\sqrt{n} \cdot e^{\log^* n}$ which entirely avoids some color. We also completely resolve the analogous question for ordinary monochromatic directed paths in general tournaments, as well as natural generalizations of the Ruzsa-Szemerédi problem which we encounter through our investigation.

Off-diagonal hypergraph Ramsey numbers
Dhruv Mubayi

A k-uniform hypergraph H with vertex set V is a collection of k-element subsets of V. We write $K_n^{(k)}$ for the complete k-uniform hypergraph with n vertices (it comprises all $\binom{n}{k}$ k-subsets of an n-element vertex set). The Ramsey number $r_k(s,n)$ is the minimum N such that every red-blue coloring of the edges of $K_N^{(k)}$ contains a monochromatic red copy of $K_s^{(k)}$ or a monochromatic blue copy of $K_n^{(k)}$. Due to its wide range of applications in logic, number theory, analysis, geometry, and computer science, estimating Ramsey numbers has become one of the most central problems in combinatorics. The *off-diagonal* Ramsey number, $r_k(s,n)$ with k,s fixed and n tending to infinity, has been extensively studied since the 1940's. Erdős and Hajnal proved in 1972 that

$$r_k(s,n) \geq \mathrm{twr}_{k-1}(\Omega(n)),$$

for $k \geq 4$ and for all $s \geq 2^{k-1} - k + 3$ where the tower function $\mathrm{twr}_k(x)$ is defined by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1} = 2^{\mathrm{twr}_i(x)}$. They conjectured that the same result holds for all $s \geq k+1 \geq 5$. Conlon, Fox, and Sudakov proved this for all $s \geq \lceil 5k/2 \rceil - 3$. The following result nearly settles the Erdős-Hajnal Conjecture.

Theorem. *There is a positive constant $c > 0$ such that for $k \geq 4$ and $n > 3k$, we have $r_k(k+3,n) \geq \mathrm{twr}_{k-1}(cn)$.*

There are two novel ingredients to our constructions. First, we relate these problems to estimates for Ramsey numbers of ordered tight-paths versus cliques, a problem that goes back to the 1930's. Second, we use $(k-1)$-uniform diagonal Ramsey numbers for more than two colors to obtain constructions for k-uniform off-diagonal Ramsey numbers for two colors. This differs from the usual paradigm in this area, exemplified by the stepping up lemma, where the number of colors stays the same or goes up as the uniformity increases.

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