Probability on trees and planar graphs: final report

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1 Overview of the subject

STATISTICAL PHYSICS IN TWO DIMENSIONS

Statistical physics in two dimensions has been one of the hottest areas of probability in recent years. The introduction of Schramm's SLE and further developments by Lawler, Schramm and Werner on the one hand, and the application of discrete complex analysis, as pioneered by Smirnov, on the other, have led to several breakthroughs and to the resolution of a number of long-standing conjectures. These include the conformally invariant scaling limits of critical percolation and Ising models, and the determination of non-intersection and other critical exponents and dimensions of sets associated with planar Brownian motion (such as the frontier and the set of cut points). It is manifest that much progress will follow, possibly including the treatment of self-avoiding walks, the $O(n)$ loop model and the Potts model. While the bulk of this body of work applies to specific lattices, there are many fascinating problems in extending results to more general planar graphs.

One natural next step is to study the classical models of statistical physics in the context of random planar maps. There are deep conjectured connections between the behaviour of the models in the random setting versus the Euclidean setting, most significantly the KPZ formula of Knizhnik, Polyakov and Zamolodchikov from conformal field theory. This formula relates the dimensions of certain sets in Euclidean geometry to the dimensions of corresponding sets in the random geometry. Establishing this connection rigorously may provide a systematic way to analyze models on the two dimensional Euclidean lattice: first study the model in the random geometry setting, where the Markovian properties of the underlying space make the model tractable; then use the KPZ formula to translate the critical exponents from the random setting to the Euclidean one. Several models, notably the self avoiding walk and the $O(n)$ model are much easier to analyze in the setting of random geometry. As mentioned before, much of this picture is conjectural. However, with our rapidly growing understanding of the structure of random planar maps and discrete complex analysis, there is hope of making progress in this direction.

RANDOM MAPS

Random planar maps is a widely studied topic at the intersection of probability, combinatorics and statistical physics. There are many natural classes of finite planar maps from which one can draw a random element: triangulations or p-angulations, maps of more general face size, maps of higher genus, and even the set of all planar graphs on n vertices. As in the case of random walks, where the large scale behaviour is invariant to the microscopic properties of the walk (it converges to Brownian motion), it is expected that random planar maps exhibit a similar universality (with the exception of random trees, which lie in a different universality class: see the last section of this proposal).

Let us consider a concrete example. Fix $p \geq 3$, and let T_n be a random p-angulation of the sphere $\mathbb{S}^2$, chosen uniformly among such all p-angulations with n vertices (considered up to orientation-preserving homeomorphism). In the recent culmination of a series of papers, Le Gall and (independently) Miermont show that when $p = 3$ or when $p \geq 4$ is even, the random graph metric of T_n scaled by $n^{-1/4}$ converges (in the Gromov–Hausdorff sense) to a random compact metric space homeomorphic to the sphere. This limiting metric has been termed the *Brownian Map*.

A second approach to the investigation of random maps is to study the unscaled or "infinite volume" limit of random maps. One such limit, known as the uniform infinite planar triangulation (UIPT), was shown to exist by Angel and Schramm, and is obtained by taking the local limit of uniform random finite triangulations; other models of maps possess similar limits. The research in this area is focused on almost sure geometric properties of the limiting object: it is an invariant, planar, recurrent, and polynomially growing graph (the volume growth exponent is 4), with a very fractal geometry. In particular, the random walk on it exhibits anomalous diffusion. Many questions about the UIPT and its (unfortunately named) counterpart, the stochastic hyperbolic infinite triangulation, remain open; for example, the speed of the random walk on the UIPT is conjectured to be $n^{1/4}$.

All the research described above is concerned with the graph distance of random maps. However, planar maps can also be viewed as manifolds and, as such, are endowed with a conformal structure. For example a triangulation gives rise to a Riemann surface by making each triangle equilateral. In particular, there are natural embeddings of these maps in the sphere or the plane, given by the Riemann uniformization of the conformal structure.

There are also discrete analogues of this embedding based on tools from discrete conformal geometry, and we can find embeddings using circle packings (see next section), rubber bands or square tilings. Understanding how these embeddings behave can shed light on many models of statistical physics on random planar maps. For instance, a concrete conjecture states that the empirical mass measure of the circle packing of a random triangulation on n vertices (that is, the measure giving each circle mass $1/n$) converges to *Liouville quantum gravity*, a continuous model of random surfaces developed by Duplantier and Sheffield. Proving this conjecture will advance us considerably towards the goals presented in the previous section.

CIRCLE PACKING AND RANDOM WALKS ON PLANAR GRAPHS

An important tool in the study of planar maps is the theory of circle packing, which provides a canonical and conformally natural way to embed a planar graph into the plane. A circle packing is a collection of circles in the plane with disjoint interiors. The tangency graph of a circle packing is a planar graph in which the vertex set is the set of circles and two circles are neighbours if they are tangent in the packing. It is clear that the tangency graph is planar. Conversely, the celebrated Koebe–Andreev–Thurston Circle Packing Theorem states that any finite planar graph can be realized as a tangency graph of a circle packing. Many extensions and generalizations of this theorem are known.

This beautiful theory has a wide range of applications, in combinatorics, computer science, differential geometry, geometric analysis, and complex analysis as well as discrete probability. In particular, circle packings have been indispensable for the analysis of random walks on planar graphs.

An example that highlights the connection to probability is a seminal theorem of He and Schramm which states that a bounded degree infinite triangulation is transient if and only if it can be circle packed in a bounded domain. This bounded circle packing gives rise to a natural definition of a geometric "boundary" of a transient planar graph. A natural problem is to understand the relation between this and other notions of boundary, such as the Poisson and Martin boundaries. A deep result in this area is that the Poisson boundary contains the geometric boundary. In the context of (continuous) manifolds, these questions have been central in geometric analysis, since the work of Yau, and are very difficult. The additional assumption of planarity provides us with a rich set of ideas and tools.

RANDOM TREES AND CRITICAL PERCOLATION

Random trees have a long and illustrious history in the combinatorics literature. However, work on their scaling limits is a much more recent development which was initiated in a sequence of seminal papers by Aldous. The prototypical result is that the random graph metric obtained from drawing a random uniform tree from the set of n^{n-1} rooted trees on n labelled vertices converges (again, in the Gromov–Hausdorff sense), when rescaled by $n^{-1/2}$, to a random compact metric space. This is the so-called *Brownian continuum random tree* (BCRT).

This topic is more mature than the topic of random planar maps and the universality phenomenon for random trees is much better understood. In particular, the BCRT is known to be the limit for a large class of random trees: critical Galton–Watson trees whose offspring distributions have finite variance, unordered binary trees, uniform unordered trees, critical multitype Galton–Watson trees and many others. Moreover, the techniques used to prove convergence to the BCRT have also been adapted to give new limits each having its own domain of attraction: the stable trees (the scaling limits of critical Galton–Watson trees whose offspring distribution are in the domain of attraction of a stable law of index in $(1, 2)$), Lévy trees, fragmentation trees (which are the scaling limits of the so-called Markov branching trees) and the minimum spanning tree of the complete graph.

The BCRT is now recognized as central to the scaling limits of many discrete and highly varied objects. For example, the scaling limit of critical Erdős–Rényi random graphs, has been shown by Addario-Berry, Broutin and Goldschmidt to be composed of a number of rescaled BCRT's glued together. Excitingly, there is evidence that this limiting object is a universal limit for a wide variety of models including random graphs with a fixed degree sequence, percolation on random regular graphs, critical percolation on high-dimensional tori and on the hypercube, the critical vacant set left by random walk on a random regular graph, and the SIR epidemic model. The Brownian map discussed above is also defined as a certain quotient of the BCRT. This fruitful and expanding area of study should yield interesting developments for many years to come.

2 Overview of the workshop

During the week of the workshop we heard many superb lectures on the topics above, we also had two spontaneous simulation sessions (where participants demonstrated some computer simulations) and a comprehensive problem session. All this stimulated much work and many discussions. What follows is a summary of some of the substantial open problems that were presented during the workshop. Feedback from participants was enthusiastic. For example, Yuval Peres wrote:

> This was a superb and very timely workshop. Great progress is being made in this area, where several strands of thought (Mathematical Models of quantum gravity, circle packings, planar maps and random walks) are showing deeper connections than ever before. The excitement was tangible during many of the talks.
>
> This will certainly have a long term effect on my research, and I am sure this holds for most of the participants.

3 Overview of some of the open problems arising from or presented at the workshop.

As noted, a large number of open problems were presented, sparking discussions among participants.

MACROSCOPIC CIRCLES IN THE CIRCLE PACKING OF THE UIPT?
Ori Gurel-Gurevich

Consider the circle packing of the UIPT. It is a circle packing covering the plane without accumulation points (since the UIPT is one-ended almost surely). Normalize so that the circle of the origin ρ of the UIPT is centred at the origin and of radius 1. A circle in this packing is called **macroscopic** if its radius is comparable to its distance from the origin.

Question. *Is it true that almost surely there are only finitely many macroscopic circles?*

A positive answer would enable us to gain much progress towards further rigorous understanding of the KPZ correspondence and other various properties of the SRW on the UIPT (in particular, this would imply that the distance exponent is $1/4$).

BOUNDARY OF THE UIHPT
Nicolas Curien

Consider the uniform infinite half-planar triangulation (with root-edge in the boundary). Consider the uniformization of the map (i.e. view it as a Riemann surface by putting the metric of an equilateral triangle inside each face) with the root-edge mapped to the interval $[0, 1]$. Now focus on the boundary, and rescale it in such a way that the first n edges counted to the right of 0 are mapped onto $[0, 1]$. Put mass $1/n$ at every vertex in the boundary. This yields a random measure μ_n on $\mathbb{R}$ such that $\mu_n((0, 1]) = 1$. It is conjectured that, in the limit, $\mu_n \to \mu$, where μ is the exponential of a GFF plus a log-singularity (see Sheffield (2010)).

Independently of μ, sample an SLE_6 process $(\gamma_t)_{t\geq 0}$ in the upper half-plane and let $T = \{t \in \mathbb{R}_+ : \gamma_t \in \mathbb{R}\}$. Consider $(t; \gamma_t; \mu([0, \gamma_t]))_{t\in T}$, up to time-reparametrization (taking as a convention that $\mu([0, -2]) = -\mu([-2, 0])$). Under the $(\star)$ assumption of Curien (2014), any subsequential limit μ will satisfy

$$(t; \mu([0, \gamma_t]))_{t\in T} \overset{\text{(d)}}{=} \left(t; \begin{cases} S_t^+ & \text{if } t \in \tau^+ \\ -S_t^- & \text{if } t \in \tau^- \end{cases} \right)_{t\in\tau^+\cup\tau^-} \tag{1}$$

(up to time-reparametrization), where S^+ and S^- are independent $(3/2)$-stable processes with only negative jumps, τ^+ is the set of record times of new minima for S^+ and τ^- is the set of record times of new minima for S^-.

Question. *Does the law of the LHS of (1) characterize the law of μ?*

This is of interest because, by results of Duplantier, Miller and Sheffield (2014), it is known that if instead of μ we take a GFF with the correct log-singularity, its law is indeed characterized by the corresponding quantity. If the answer to the question is in the affirmative, and the (currently unproven) $(\star)$ condition holds, then this would prove the conjecture that μ is the exponential of a GFF plus the log-singularity.

A similar conjecture exists in the upper half-plane.

A somewhat weaker version for SLE specialists: suppose you are given a random variable X (of unknown law) and, independently, sample an SLE_6 in a box of height 1 and width X, started from the top-left-hand corner. Let N_X be the number of crossings of the SLE between the top and bottom of the box.

Question. *Given the law of N_X, can one recover the law of X?*

k-DEPENDENT COLOURINGS
Omer Angel

Recently Holroyd and Liggett resolved a problem that fascinated many people for some years, by showing that there is a stationary, proper 4-colouring of $\mathcal{Z}$, which is 1-dependent (i.e. for any two sets $S, T \subset \mathcal{Z}$ with $d(S,T) > 1$, the colours in S are independent of the colours in T. Holroyd and Liggett also find 2-dependent 3-colouring of $\mathcal{Z}$, and 1-dependent q-colouring of $\mathcal{Z}^d$ for some q. In particular, these constructions provide new and more natural examples of processes which are k-dependent but not a block factor of i.i.d. variables. Many questions remain open around these strange colourings.

Question. *What is the minimal number q of colours so that a k-dependent q-colouring of $\mathcal{Z}^d$ exists?*

Question. *Is there a q-dependent k-colouring of the d-regular tree, for some q, k?*

THE GINIBRE ENSEMBLE AND THE GFF
Nathanaël Berestycki

Consider the Ginibre ensemble, that is an $n \times n$ random matrix in which all entries are independent complex Gaussians of mean zero and variance $1/n$. Let p_n be the characteristic polynomial and let $h_n(z) = \log|p_n(z)| - \mathbb{E}[\log|p_n(z)|]$. A theorem of Rider and Virág (2007) states that $h_n \to h$ weakly as $n \to \infty$, where h is a planar Gaussian free field conditioned to be harmonic outside the unit disk.

Question. *Is it true that a rescaled version of $|p_n(z)|$, viewed as a measure, converges to e^h, a Liouville quantum gravity measure? Is this related or not to the enumeration of planar maps via matrix integrals?*

LIOUVILLE BROWNIAN MOTION
Nathanaël Berestycki

Take $(B_t, t \geq 0)$ a Brownian motion in a domain $D \subseteq \mathbb{R}^2$, h an independent Gaussian free field on D and h_ε a suitably mollified version thereof. Let $\gamma > 0$ and consider the limit time change

$$\phi(t) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_0^t e^{\gamma h_\varepsilon(B_s)} \mathrm{d}s.$$

Consider the Liouville Brownian motion, $Z_t = B_{\phi^{-1}(t)}$ and its first exit time from the ball of radius 1 around the origin,

$$\tau = \inf\{t : Z_t \notin B(0,1)\}.$$

Question. *What is the behaviour of $\mathbb{P}(\tau > t)$ as $t \to \infty$ (in the annealed setting, so that we average over the GFF as well as the law of the Brownian motion)?*

It is known that $\mathbb{E}[\tau^q] < \infty$ if and only if $q < 4/\gamma^2$; we would like to be able to say that

$$\mathbb{P}(\tau > t) \sim Ct^{-\frac{4}{\gamma^2}} \text{ as } t \to \infty.$$

Observe that if $\tau_{\text{BM}} = \inf\{t : B_t \notin B(0,1)\}$ then

$$\tau = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_0^{\tau_{\text{BM}}} e^{\gamma h_\varepsilon(B_s)} \mathrm{d}s.$$

Question. *If μ_γ is the Liouville quantum gravity measure, is τ comparable to $\mu_\gamma(B(0,1))$?*

A fact due to Barral and Jin is that $\mathbb{P}(\mu_\gamma(B(0,1)) > t) \sim Ct^{-\frac{4}{\gamma^2}}$, but their ingenious proof does not seem easily replicable.

EDGE-FLIPS IN A TRIANGULATION
Ken Stephenson

The edge flip dynamics on triangulations of size n chooses each edge at rate 1, and (if possible) removes it and adds the other diagonal of the resulting quadrangle. Little is known about this Markov chain.

Question. *What is the mixing time of this Markov chain as a function of n?*

The speaker demonstrated using his CirclePack software and issued a general invitation to study edge-flips in triangulations.

SURPRISE PROBABILITIES IN LAZY SRW ON FINITE GRAPHS
Yuval Peres

Consider a lazy simple random walk on a finite graph on n vertices and let τ_y be the hitting time of a vertex y. What is the best possible upper bound for the "surprise probabilities" $P_x(\tau_y = t)$ in terms of n and t? It can be shown that in any Markov chain

$$P_x(\tau_y = t) \leq \frac{n}{t}\,.$$

Norris, Peres and Zhai proved that for the lazy random walk one has

$$P_x(\tau_y = t) \leq \frac{4e\log n}{t}\,.$$

Question. *Can one prove $P_x(\tau_y = t) \leq \frac{C\sqrt{\log n}}{t}$ for general MArkov chains?*

There is a construction where this is sharp.

In a similar spirit, let $p^*(x,y) = \max_t p^t(x,y)$ for the lazy SRW on a graph. It is known (Norris, Peres and Zhai) that

$$\sum_y p^*(x,y) \leq C\log n\,.$$

This is sharp as seen by taking the path of length n. Is following strengthening true:

Question. *Is it true that for any Markov chain*

$$\sum_y \sum_t |p^{t+1}(x,y) - p^t(x,y)| \leq C\log n\,?$$

POSITIVE SPEED IN UNIMODULAR NON-AMENABLE GRAPHS?
Tom Hutchcroft

It is well known that on any non-amenable graph with bounded degrees the speed of the simple random walk is positive (with respect to the graph distance). This statement fails without the requirement of bounded degree: take the graph of $\mathbb{N}$, add 2^n edges between vertices n and $n+1$ and a single edge between n and $2n$. In this example the SRW after T steps will be roughly at vertex T, but the graph distance from the origin is of order $\log T$.

Question. *Let (G, ρ) be a unimodular or a stationary random graph that is a.s. non-amenable, does the simple random walk have positive speed?*

Another related question was asked during this presentation by Yuval Peres:

Question. *If (G, ρ) is a unimodular (or a stationary) random graph that is almost surely recurrent, is it true that two independent random walkers collide infinitely often? That is, the number of n's such that $X_n = Y_n$ is infinite.*

There are examples of recurrent graphs where this does not hold, but they are very far from stationary (e.g. the 2 dimensional comb graph, see Peres and Krishnapur).

SQUARINGS OF A SQUARE
Louigi Addario-Berry

A *squaring of a rectangle* is a tiling of a rectangle by squares: in other words, the squares have disjoint interiors and their union is the entire rectangle. For concreteness, consider rectangles of height 1 whose lower-left corner is at the origin in $\mathbb{R}^2$. Write s_n for the number of squarings using exactly n squares. How does s_n grow as n becomes large? This basic enumerative question remains unanswered and, perhaps surprisingly, almost unstudied.

Say that a squaring of a rectangle is *perfect* if it contains no non-trivial sub-squaring of a rectangle (each single square is a trivial sub-squaring). Write r_n for the number of perfect squarings with n squares. Tutte (1963) conjectures that

$$r_n = (1 + o(1)) \frac{4^n}{243\pi^{1/2} n^{5/2}};$$

establishing this would likely be a useful step towards finding asymptotics for s_n, and would be interesting in its own right. The conjecture for the growth of p_n is based on a classic construction of Brooks, Smith, Stone and Tutte (1940), which builds perfect squarings of rectangles from 3-connected planar graphs. The construction is neither invertible nor size-preserving, but if it is "almost invertible" and "almost always size-preserving" then the number of planar maps and the number of perfect squarings should be asymptotically equal.

The construction mentioned above works as follows. Let G be a 3-connected planar graph, and fix an oriented edge st of G. View G as an electrical network in which each edge is a unit resistor. Remove edge st, put potential 1 at s and ground at t. The squaring corresponding to G is constructed by creating a square s_e for each edge e of G; the side length of s_e is precisely the current flowing through e in the electrical network. The horizontal and vertical positions of the squares are determined (uniquely) by using Kirchoff's laws: the vertical position of the top s_e is given by the higher-potential endpoint of e. Horizontal positions are determined similarly, using the planar dual G^* of G.

In the above construction, a square s_e may degenerate to a point if no current flows along edge e in the network. This is the reason that the construction is not bijective and is not always size-preserving (there may be fewer squares in the squaring than there are edges in $G - st$). As a step towards solving the above enumerative questions, one might therefore study the following. Let (G, st) be a uniformly random 3-connected planar graph with n edges, together with an oriented edge of G. Viewed as an electrical network, what the probability p_n that some edge of G has zero current? In particular, does $p_n \to 0$ as $n \to \infty$?

GREEN FUNCTION OF RWS WITH MEMORY
Remco van der Hofstad

Let $p_n(x)$ denote the number of n-step SRW on Z^d from the origin to x and let $G_z(x) = \sum_{n\geq 0} z^n p_n(x)$. The critical point is $z_c = 1/(2d)$. Let $b_n(x)$ denote the number of n-step non-backtracking walks from the origin to $x \in \mathcal{Z}^d$ and denote the Green function $B_z(x) = \sum_{n\geq 0} z^n b_n(x)$. The critical point is $z_c = 1/(2d-1)$. These two functions satisfy an interesting relation:

$$B_z(x) = \frac{1-z^2}{1+(2d-1)z^2} G_{\frac{z}{1+(2d-1)z^2}}(x)\,.$$

Now consider walks of memory 2 (i.e., they are not allowed to close a square) and let $m_n(x)$ be the number of such walks of length n and $M_z(x) = \sum_{n\geq 0} z^n m_n(x)$.

Question. *What is the critical point of M_z? Is there a nice formula relating M_z, B_z and G_z?*

REMOVING SYMMETRY ASSUMPTION IN PERCOLATION
Vincent Tassion

Many methods in the study of percolation depend on symmetries of the underlying graph, even where the result should hold for more general graphs. We would like to ave methods to study percolation that are less reliant on such symmetries.

As an example, consider percolation on $\mathcal{Z} \times \mathbb{N}$, for some parameter p, and consider the events A_+, A_- that $(0, n)$ is connected respectively to the right and left halves of the boundary. By symmetry, $\mathbb{P}(A_+) = \mathbb{P}(A_-)$.

If we add a generator $(1, 1)$, then the graph becomes the triangular lattice. There is a simple argument to show that $\mathbb{P}(A_-) \geq \mathbb{P}(A_+)$, and this is based on symmetry of the triangular lattice.

Question. *Is it true that $\mathbb{P}(A_+) \leq \mathbb{P}(A_-)$ if we take the generator set $(1, 0), (0, 1)$ and some additional (x_i, y_i) with $x_i, y_i \geq 0$?*

Intuitively, adding such diagonal edges helps connectivity in the corresponding direction.