Hypercube percolation

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Hypercube percolation

Study random subgraph of hypercube $\{0,1\}^n$, with bonds

$$\{\{x,y\}\colon \exists!\ i \text{ with } x_i \neq y_i\}.$$

Make bonds $\{x,y\}$ independently

occupied	with probability p,
vacant	with probability $1-p$,

where $p \in [0,1]$ is percolation parameter.

Erdős and Spencer (1979): for $p = 1/2 + \varepsilon/(2n)$, random graph connected with probability $\mathrm{e}^{-\mathrm{e}^{-\varepsilon}(1+o(1))}$.

Our goal: Study percolation phase transition as $n \to \infty$.

Inspiration: Erdős-Rényi random graph

Erdős-Rényi random graph is random subgraph of complete graph on n vertices where each of $\binom{n}{2}$ edges is occupied with probab. p.

Phase transition: (Erdős and Rényi (60))
For $p = (1+\varepsilon)/n$, largest connected component $|\mathscr{C}_{\max}|$ has size
(a) $\Theta_{\mathbb{P}}(\log n)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(n)$ for $\varepsilon > 0$;

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1+\varepsilon)/n$, largest component size is $\Theta_{\mathbb{P}}(n^{2/3})$ with expected cluster size $\Theta(n^{1/3})$ whenever $|\varepsilon| \leq \Lambda n^{-1/3}$.

Barely sub- and supercritical regimes:
(a) $|\mathscr{C}_{\max}| = 2\varepsilon n(1 + o_{\mathbb{P}}(1))$ when $\varepsilon \gg n^{-1/3}, \varepsilon = o(1)$;
(b) $|\mathscr{C}_{\max}| = 2\varepsilon^{-2} \log(\varepsilon^3 n)(1 + o_{\mathbb{P}}(1))$ when $\varepsilon \ll -n^{-1/3}, \varepsilon = o(1)$.

BCHSS approach to criticality

Recall $\mathbb{E}_p|\mathscr{C}(0)| = \Theta(n^{1/3})$ for Erdős-Rényi random graph and $p = 1/n$, where n is volume graph.

BCHSS05a,05b,06: Define critical threshold $p_c(\{0,1\}^n)$ as solution

$$\chi(p) = 2^{n/3},$$

where $\chi(p) = \mathbb{E}_p|\mathscr{C}(0)|$ is expected cluster size on hypercube.

Challenge: Prove that $p_c(\{0,1\}^n)$ really is critical value, by investigating connected component sizes above and below $p_c(\{0,1\}^n)$.

(Sub-)Critical results

Theorem 1 (Subcritical clusters) (BCHSS (05a), (05b)).
For $p = p_c(1+\varepsilon)$, and uniformly in $\varepsilon \leq 0$, as $n \to \infty$,

$$\chi(p) = \frac{O(1)}{|\varepsilon| + 2^{-n/3}},$$

$$\mathbb{P}_p\Big(\chi^2(p) \leq |\mathscr{C}_{\max}| \leq 2\chi^2(p) \log\big(2^n/\chi^3(p)\big)\Big) \geq 1 - \log\big(2^n/\chi^3(p)\big)^{-3/2}.$$

Theorem 2 (Scaling window) (BCHSS (05a), (05b)).
For $p = p_c(1+\varepsilon)$, with $|\varepsilon| \leq \Lambda 2^{-n/3}$, there exists $b_1 = b_1(\Lambda) > 0$ s.t.

$$\mathbb{P}_p\Big(\omega^{-1} 2^{2n/3} \leq |\mathscr{C}_{\max}| \leq \omega 2^{2n/3}\Big) \geq 1 - \frac{b_1}{\omega}.$$

vdHHe (11): $|\mathscr{C}_{\max}| 2^{-2n/3}$ is tight and non-degenerate:

hallmark of critical behavior.

Percolation on high-dimensional tori

Results work on general high-dimensional tori $\mathbb{T}_{r,d}$, subject to a

finite graph triangle condition.

Define $p_c(\mathbb{T}_{r,d};\lambda)$ such that $\mathbb{E}_{p_c}|\mathscr{C}(0)| = \lambda r^{d/3}$, and let

$$\tau_p(x) = \mathbb{P}_p(0 \longleftrightarrow x).$$

Then, finite-graph triangle condition states that

$$\nabla(p_c(\mathbb{T}_{r,d};\lambda)) = \sum_{x,y} \tau_{p_c}(x)\tau_{p_c}(y-x)\tau_{p_c}(y) = 1 + o(1) + O(\lambda^3).$$

Known for

▷ Percolation on nearest-neighbor torus $\mathbb{T}_{r,d}$ for d large;

▷ Percolation on spread-out torus and $d > 6$;

▷ Hypercube percolation.

[Beware, not obvious that $p_c(\mathbb{T}_{r,d};\lambda)$ close to $p_c(\mathbb{Z}^d)$ for high-d tori...]

Supercritical results

Theorem 3 (Supercritical clusters) (vdH+Nachmias (12)).
For $p = p_c(1+\varepsilon)$, with $\varepsilon = o(1), \varepsilon \gg 2^{-n/3}$,

$$\frac{|\mathscr{C}_{\max}|}{2\varepsilon 2^n} \xrightarrow{\mathbb{P}} 1,$$

while, with $\mathscr{C}_{(2)}$ denoting second largest cluster,

$$\frac{|\mathscr{C}_{(2)}|}{\varepsilon 2^n} \xrightarrow{\mathbb{P}} 0,$$

$$\mathbb{E}_p|\mathscr{C}(0)| = 4\varepsilon^2 2^n(1 + o(1)).$$

Percolation phase transitions hypercube and complete graph alike: 2ε is asymptotic survival probability of branching process with Poisson$(1+\varepsilon)$-offspring distribution.

Proof (sub)critical regime BCHSS

(a) Differential inequalities and finite-graph triangle condition (as in Aizenman+Newman 84, Barsky+Aizenman 91);

(b) Lace expansion for two-point function (as in Hara+Slade 90) to relate finite-graph triangle condition to random walk condition;

(c) Analysis (a)-(b) proving bounds on expected and largest cluster.

$\triangleright$ Part (b) is only one that uses lace expansion.

(Later explain how lace expansion can be avoided for hypercube.)

General supercritical setting

Main result follows from general result applying to many

high-dimensional transitive graphs of large degree m.

Applies under three conditions on transitive base graph on which percolation is performed:

(1) degree m tends to infinity with volume graph;

(2) $[(m-1)p_c]^{T_{\mathrm{mix}}} = 1 + o(1)$, where T_{mix} is mixing time non-backtracing random walk;

(3) finite graph NBW triangle condition.

Proof 1: supercritical cluster tails

Theorem 4 (Supercritical cluster tails). Let $p = p_c(1+\varepsilon)$ with $\varepsilon \gg V^{-1/3}$. Then, for $k_0 = \varepsilon^{-2}(\varepsilon^3 V)^{\alpha}$ for any $\alpha \in (0, 1/3)$,

$$\mathbb{P}_p(|\mathscr{C}(0)| \geq k_0) = 2\varepsilon(1 + o(1)).$$

Further, with

$$Z_{\geq k} = \Big|\Big\{v \colon |\mathscr{C}(v)| \geq k\Big\}\Big|,$$

denoting number of vertices in large clusters,

$$\frac{Z_{\geq k_0}}{2\varepsilon 2^n} \xrightarrow{\mathbb{P}} 1.$$

Proof: Differential inequalities (as in BA91, BCHSS05a), with careful analysis of constants involved.

Proof 2: non-backtracking random walk

Let $\mathbb{P}_p(0 \overset{[a,b]}{\longleftrightarrow} x)$ denote probability that

0 is connected to x with shortest path of length $\in [a, b]$.

We bound $\mathbb{P}_p(0 \overset{[r,2r]}{\longleftrightarrow} x)$ by a constant in x for $r \gg 1/\varepsilon$ by comparing percolation paths to non-backtracking random walk (i.e., random walk conditioned not to reverse immediately).

In particular, with $V = 2^n$,

$$\mathbb{P}_p(0 \overset{[r,2r]}{\longleftrightarrow} x) \leq \frac{1 + o(1)}{V} \sum_x \mathbb{P}_p(0 \overset{[r,2r]}{\longleftrightarrow} x) \leq (1 + o(1))\mathbb{E}_p|B_0(2r)|/V,$$

where $B_0(r)$ is ball radius r in graph metric on G_p.
This makes performing complicated sums relatively easy.

Proof 3: large clusters share boundary

For $x, y \in \{0,1\}^n$ and ℓ, let

$$S_\ell(x,y) = \left|\left\{(u,u') \in E \colon x \overset{\ell}{\longleftrightarrow} u \,, y \overset{\ell}{\longleftrightarrow} u'\right\}\right| .$$

Pair x, y is good when $|\mathscr{C}(x)|, |\mathscr{C}(y)| \geq K\varepsilon^{-2}$, and

$$S_{2r+r_0}(x,y) \geq \alpha_n(nV)\Big(\mathbb{E}|B_0(r_0)|/(2\varepsilon)\Big)^2 V^{-2},$$

where $\alpha_n \to 0$ very slowly. Write $P_{r,r_0,K}$ for number of good pairs.

Theorem 5 (Most large clusters share many boundary edges).
Assume $\varepsilon^3 2^n \to \infty, \varepsilon \leq n^{-2}$. Take $r = M/\varepsilon, r_0 = \frac{1}{2}\varepsilon^{-1}\log(\varepsilon^3 2^n)[1 - o(1)]$.
Then, there exist $K, M \to \infty$ s.t.

$$\frac{P_{r,r_0,K}}{(2\varepsilon V)^2} \xrightarrow{\mathbb{P}} 1 \,.$$

Proof 4: improved sprinkling

Take $p_2 = \delta\varepsilon/n$ for our sprinkling probability, where $\delta > 0$ is small. Let p_1 satisfy

$$p = p_1 + (1 - p_1)p_2 \,,$$

Given G_{p_1}, construct auxiliary simple graph $H = (\mathcal{V}, \mathcal{E})$ with

$$\mathcal{V} = \left\{x \in G_{p_1} : |\mathscr{C}(x)| \geq K\varepsilon^{-2}\right\}, \quad \mathcal{E} = \left\{(x, y) \in \mathcal{V}^2 \text{ good pair}\right\},$$

$$\mathbb{P}_{p_1}\left(P_{r,r_0,K} \geq (1-\alpha)4\varepsilon^2 2^{2n}, |\mathcal{V}| \in [(2-\theta)\varepsilon, (2+\theta)\varepsilon]\right) \geq 1 - \delta.$$

Claim: Probability there exists partition $\mathcal{V} = M_1 \uplus M_2$ with $|M_1| \geq \theta|\mathcal{V}|$ and $|M_2| \geq \theta|\mathcal{V}|$ s.t. sprinkled edges do not connect M_1 to M_2 is small.

Step 1: Number of partitions is at most $2^{3K^{-1}\varepsilon^3 V}$.

Step 2: Given such partition, number of edges (u, u') between $\mathscr{C}(M_1)$ and $\mathscr{C}(M_2)$ is at least $\alpha_n^3\varepsilon^2 n 2^n$ for some large K. [Cheat]

Proof 4: improved sprinkling (Cont.)

▷ Recall that $p_2 = \delta\varepsilon/n$ is sprinkling probability, with $\delta > 0$ small.

▷ Recall that number of edges (u, u') between $\mathscr{C}(M_1)$ and $\mathscr{C}(M_2)$ is at least $\alpha_n^3 \varepsilon^2 n 2^n$ for some large K.

Conclusion: Probability sprinkling fails is at most

$$2^{3K^{-1}\varepsilon^3 V}\Big(1 - \delta\varepsilon/n\Big)^{\alpha_n K^{-2}\varepsilon^2 n 2^n} \leq 2^{3K^{-1}\varepsilon^3 V} \mathrm{e}^{-\alpha_n^3 \delta \varepsilon^3 2^n},$$

which converges to zero rapidly when $\varepsilon^3 2^n \gg 1$ and $K\alpha_n^3 \gg 1$.

Can achieve this by letting $\alpha_n \to 0$ and $K \to \infty$ sufficiently slowly. For example, $K = (\varepsilon^3 2^n)^{1/4}, \alpha_n = (\varepsilon^3 2^n)^{-1/13}$ will do.

Proof 5: Unlacing hypercube percolation

For hypercube, we aim to show that

$$[(n-1)p_c]^{T_{\text{mix}}} = 1 + o(1),$$

where T_{mix} is mixing time non-backtracing random walk, which equals $T_{\text{mix}} = (1+o(1))n\log(n)$ (vdH-Fitzner 14).

This amounts to showing $\mathbb{E}_p|\mathscr{C}(0)| \gg 2^{n/3}$ for $p = 1/(n-1) + K/n^3$ with $K > 0$ large.

Allows us to identify phase transition hypercube without relying on lace expansion.

Lemma. For $p = (1 + 5/(2n^2) + K/n^3)/(n-1)$ with K sufficiently large, as long as $\mathbb{E}_p|B(k)| \leq 2^{n/2}$, there exists $c > 0$ s.t.

$$\mathbb{E}_p|\partial B(k)| \geq [1 + c/n^3]\mathbb{E}_p|\partial B(k-1)|.$$

High-dimensional tori

Theorem 5. (Heydenreich-vdH (07-11)) For nearest-neighbour percolation in sufficiently high dimensions, or for sufficiently spread-out percolation in $d > 6$,

$$p_c(\mathbb{Z}^d) = p_c(\mathbb{T}_{r,d}; \lambda) + O(r^{-d/3}).$$

As a result, all results from BCHSS05a apply.

Proof uses subtle coupling of clusters on torus and $\mathbb{Z}^d$, similar in spirit to that by Benjamini-Schramm (96), that shows that

$$|\mathscr{C}_{\mathbb{T}}(0)| \preceq |\mathscr{C}_{\mathbb{Z}}(0)|,$$

together with an accompanying lower bound.

Progress nearest-neighbor model

Theorem 6. (Fitzner-vdH (14?)) Nearest-neighbour percolation displays mean-field critical exponents for $d \geq 12$.

Proof by Hara-Slade was never published, so project improves transparancy in high-dimensional percolation.
Mathematica code can be downloaded from website Robert Fitzner, for everyone to check and play with.

Proof uses
(a) Non-backtracking lace expansion (NoBLE) taking more of interaction explicitly into account;
(b) sharp bounds on NoBLE coefficients;
(c) analysis of NoBLE assuming explicit numerical conditions;
(c) a computer-assisted proof that verifies these conditions.

Open problems and extension

(a) Study limit in probability of largest cluster in subcritical phase on hypercube.

(b) Study limit in probability of second largest component in super-critical phase on hypercube, and prove discrete duality principle.

(c) Prove CLT for giant component.

(d) Show that scaling limit of largest clusters is same (modulo trivial scalings) as that for ERRG in Aldous (97).

(e) Extension percolation on torus to nearest-neighbor percolation with $d \geq 12$.

Literature hypercube percolation

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