An Active-Set Quadratic Programming Method Based On Sequential Hot-Starts

Andreas Wächter

Department of Industrial Engineering and Management Sciences
Northwestern University
waechter@iems.northwestern.edu

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Outline

- Solving sequences of "similar" QPs
- Goal: Use information computed for first QP to accelerate the (approximate) solution of later QPs
  - Avoid factorization of matrices
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  - Iterative refinement
  - Accelerated iterative linear solver
- Numerical results
  - Model predictive control
  - Sequence of random NLPs

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- In collaboration with:
  - Christian Kirches
  - Travis Johnson

Solve Sequence of Similar Quadratic Programs

$$QP$$

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & \tfrac{1}{2} d^T W d + g^T d \\ \text{s.t.} \quad & A d + c = 0 \\ & d \geq l \end{aligned}$$

- $W \in \mathbb{R}^{n\times n}$ (pos.def.), $A \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, and $l \in \mathbb{R}^n$

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QP_0

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QP_1

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QP_2

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & \tfrac{1}{2} d^T W_2 d + g_2^T d \\ \text{s.t.} \quad & A_2 d + c_2 = 0 \\ & d \geq l_2 \end{aligned}$$

$\ldots$

- Solve sequence of QPs
  - Same size, with "similar" data: $W_0 \approx W_i$, $A_0 \approx A_i$, $g_0 \approx g_i$, $\ldots$
- Goal: Exploit information computed for QP_0 to solve QP_i approx.

Example: Sequential Quadratic Programming

NLP

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & c(x) = 0 \\ & x \geq 0 \end{aligned}$$

$\longrightarrow$

QP_k

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & \tfrac{1}{2} d^T W_k d + g_k^T d \\ \text{s.t.} \quad & A_k d + c_k = 0 \\ & d \geq l_k \end{aligned}$$

- Solve QP for each iterate x_k
  - $W_k \approx \nabla^2 \mathcal{L}(x_k, \lambda_k)$, $A_k = \nabla c(x_k)^T$, $g_k = \nabla f(x_k)$, $c_k = c(x_k)$, $l_k = -x_k$

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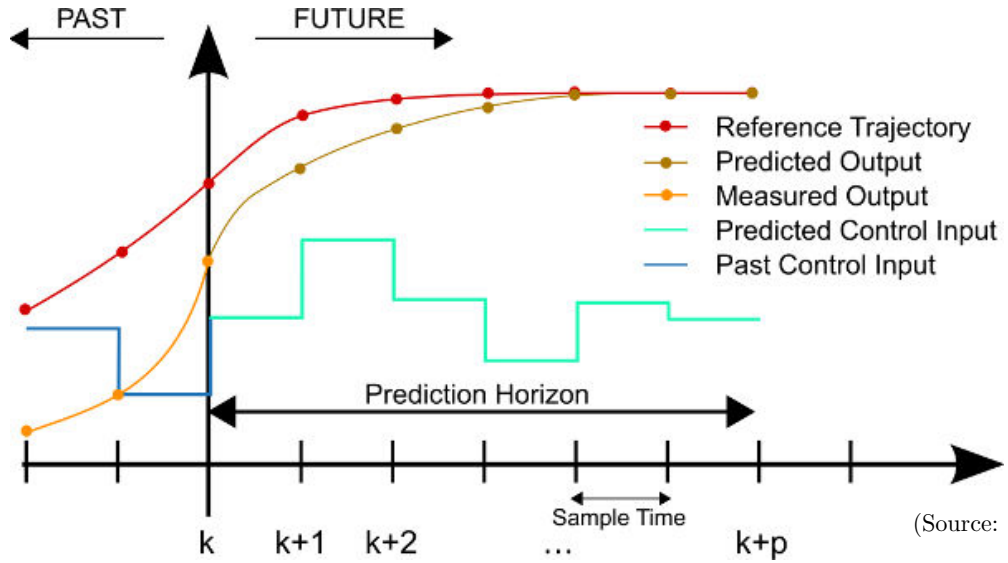
  - Example: Mixed-Integer Nonlinear Programming
    - ★ Solve nodes in a branch-and-bound tree
    - ★ Linear: dual simplex method can rapidly solve new instance
    - ★ Reason: an existing factorization of basis matrix can be used!

Example: Model Predictive Control



(Source: Martin Behrent)

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  - Optimize control over prediction horizon (NLP or QP)
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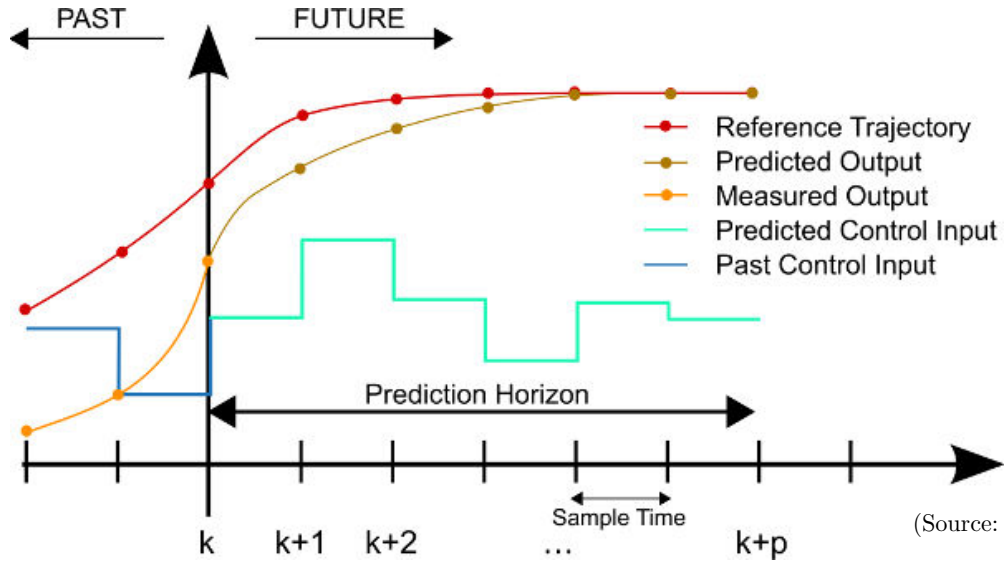

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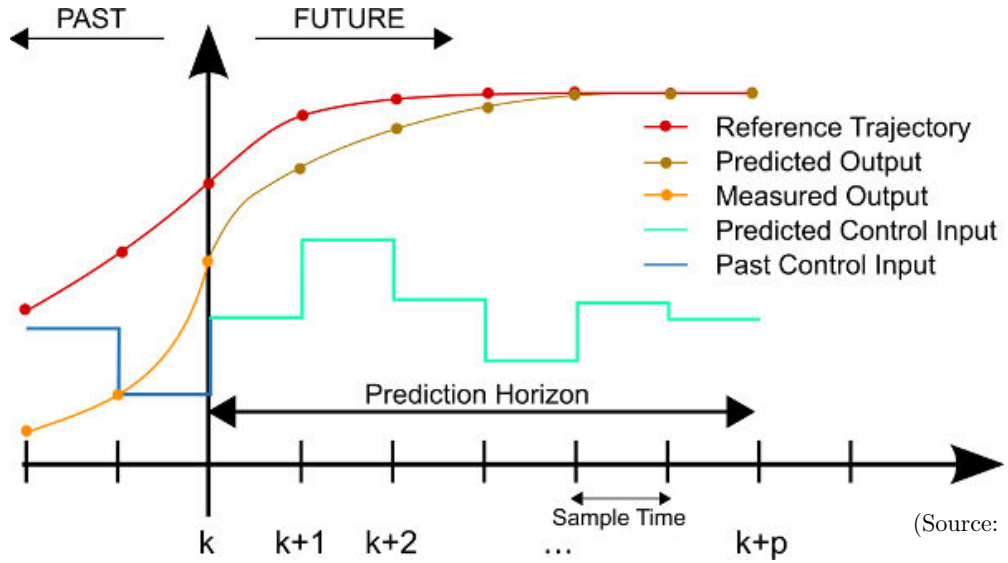

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  - Usually, little change in optimization problem
- Multiple shooting approach:
  - Integrate differential equations over subintervals $[t_k, t_{k+1}]$
  - QP data expensive
    - ★ Computation of full matrices very expensive (Av and $A^T v$ available)

Active Set QP Solver

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Compute solution from linear system:

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$\mathcal{A} \subseteq \{1, \ldots, n\}$: Guess of the active set, i.e., $d^{\mathcal{A}} = l^{\mathcal{A}}$

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$\mathcal{A}$ is correct, if $d \geq l$ and $\mu^{\mathcal{A}} \geq 0$

Warm Start vs. Hot Start

- Warm Start:

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \quad & \tfrac{1}{2} d^T \widetilde{W} d + \widetilde{g}^T d \\ \text{s.t.} \quad & \widetilde{A} d + \widetilde{c} = 0 \\ & d \geq \widetilde{l} \end{aligned} \longrightarrow \begin{aligned} \min_{d\in\mathbb{R}^n} \quad & \tfrac{1}{2} d^T W d + g^T d \\ \text{s.t.} \quad & A d + c = 0 \\ & d \geq l \end{aligned}$$

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- Appropriate algorithm: Parametric QP solver

Iterative Refinement

- Want to solve linear system

$$\boxed{M\,x + b = 0}$$

- At iterate x_i:

$$M\,x_i + b = r_i \neq 0$$

- New iterate $x_{i+1} = x_i + p_i$ should satisfy

$$M(x_i + p_i) + b = 0$$

- Get step p_i from

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- Converges if $\|I - \widetilde{M}^{-1}\,M\| < 1$.

Equality Constrained QP

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Iterative refinement ($d_{i+1} = d_i + p_i$ and $\lambda_{i+1} = \lambda_i + p_i^\lambda$)

$$\begin{bmatrix} \widetilde{W} & \widetilde{A}^T \\ \widetilde{A} & 0 \end{bmatrix} \begin{pmatrix} p_i \\ p_i^\lambda \end{pmatrix} = - \left[\begin{pmatrix} W d_i + A^T \lambda_i \\ A d_i \end{pmatrix} + \begin{pmatrix} g \\ c \end{pmatrix} \right]$$

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Inequality Constraints

Solve

$$\begin{aligned} \min_{d} \quad & \tfrac{1}{2} d^T W d + g^T d \\ \text{s.t.} \quad & A d + c = 0 \end{aligned}$$

by computing a sequence of iterative refinement steps from

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Note: We do not need iterates for bound multipliers

Inequality Constraints and Iterative Refinement

- Uses hot-starts (starts from existing factorization)
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- Key consequence: QP solver is responsible for handling active set
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  - can use your favorite hot-startable QP solver
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- Related method [Bock et al., 07]: Only one refinement step
- Disadvantage: Iterative refinement converges slowly

Accelerated Version

- Monitor active set $\mathcal{A}_i = \{j : d_i^{(j)} = l^{(j)}\}$.
- If $\mathcal{A}_i = \mathcal{A}_{i-1} =: \mathcal{A}$ during iterative refinement, set $\mathcal{F} = \mathcal{A}^C$.
- Active-set QP solver sets $d^{\mathcal{A}} = l^{\mathcal{A}}$ and solves

$$\begin{bmatrix} W^{\mathcal{F}\mathcal{F}} & (A^{\mathcal{F}})^T \\ A^{\mathcal{F}} & 0 \end{bmatrix} \begin{pmatrix} d^{\mathcal{F}} \\ \lambda \end{pmatrix} = - \begin{pmatrix} \tilde{g}^{\mathcal{F}} \\ \tilde{c} \end{pmatrix}$$

- Apply iterative linear solver (e.g., SQMR)

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- Preconditioner requires solutions of

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Could solve

$$\begin{aligned} \min_z \quad & \tfrac{1}{2} z^T \widetilde{W} z + (-r_g^{\mathcal{F}})^T z^{\mathcal{F}} \\ \text{s.t.} \quad & \widetilde{A} z - r_c = 0 \\ & 0 \le z^{\mathcal{A}} \le 0 \end{aligned}$$

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Guessed active set $\mathcal{A}$ might be too small

- As SQMR converges, we might have $d_*^{\mathcal{F}} \not\geq l^{\mathcal{F}}$
  - Stop SQMR if $d_i \not\geq l$, go back to iterative refinement.

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We solve

$$\begin{aligned} \min_z \quad & \tfrac{1}{2} z^T \tilde{W} z + (-r_g^{\mathcal{F}})^T z^{\mathcal{F}} + \left(W^{\mathcal{A},\cdot} d_i + (A^{\mathcal{A}})^T \lambda_i + g^{\mathcal{A}} \right)^T z^{\mathcal{A}} \\ \text{s.t.} \quad & \tilde{A} z - r_c = 0 \\ & 0 \le z^{\mathcal{A}} \end{aligned}$$

- Guessed active set $\mathcal{A}$ might be too large
  - Stop SQMR if $z^{(j)} > 0$ for some $j \in \mathcal{A}$

Preconditioner QP

$$\begin{aligned} \min_z \quad & \tfrac{1}{2} z^T \widetilde{W} z + (-r_g^{\mathcal{F}})^T z^{\mathcal{F}} \\ \text{s.t.} \quad & \widetilde{A} z - r_c = 0 \\ & 0 \le z^{\mathcal{A}} \le 0 \end{aligned}$$

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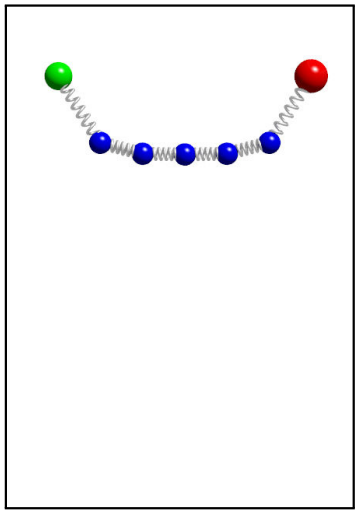
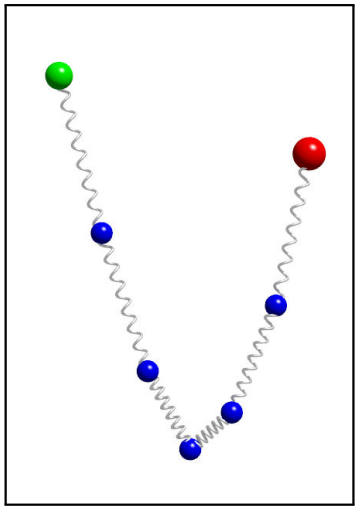
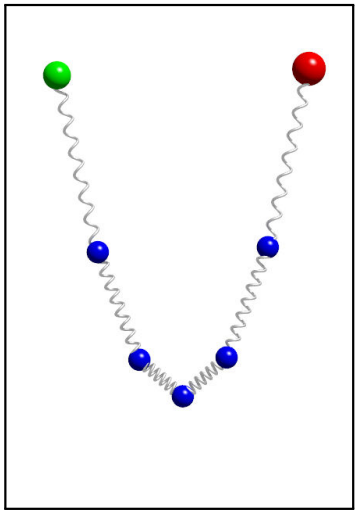
$$\begin{aligned} \min_z \quad & \tfrac{1}{2} z^T \tilde{W} z + (-r_g^{\mathcal{F}})^T z^{\mathcal{F}} + \left(W^{\mathcal{A},\cdot} d_i + (A^{\mathcal{A}})^T \lambda_i + g^{\mathcal{A}} \right)^T z^{\mathcal{A}} \\ \text{s.t.} \quad & \tilde{A} z - r_c = 0 \\ & 0 \le z^{\mathcal{A}} \end{aligned}$$

- Guessed active set $\mathcal{A}$ might be too large
  - Stop SQMR if $z^{(j)} > 0$ for some $j \in \mathcal{A}$
- Theorem [Johnson, Kirches, W 13]:
  - If iterates converge, then to optimal solution of QP_k
  - If SOSC holds, convergence to solution (d_*, λ_*) during SQMR, if (d_0, λ_0) sufficiently close to (d_*, λ_*)

Numerical Results Model Predictive Control

Controlling hanging chain

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\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & \int_0^T w_{\mathrm{v}} \sum_{i=1}^{N_{\mathrm{PM}}} \|v_i(t)\|_2^2 + w_{\mathrm{x}} \|x_{N_{\mathrm{PM}}}(t) - x_{\mathrm{e}}\|_2^2 + w_{\mathrm{u}} \|u(t)\|_2^2 + w_{\mathrm{sl}} \|u_{\mathrm{sl}}(t)\|_2^2 \, \mathrm{d}t \\
\text{s.t.} \quad & \dot{x}_i(t) = v_i(t) && t \in [0,T],\ 1 \le i < N_{\mathrm{PM}} \\
& \dot{v}_i(t) = (F_{i+1}(t) - F_i(t)) \cdot N_{\mathrm{PM}}/m - g && t \in [0,T],\ 1 \le i < N_{\mathrm{PM}} \\
& \dot{x}_{N_{\mathrm{PM}}}(t) = u(t) && t \in [0,T] \\
& x(0) = \hat{x}_0 \\
& u(t) \in [-1,1]^3 && t \in [0,T] \\
& x_1^{\mathrm{low}} \le x_1(t) \le x_1^{\mathrm{high}}, \quad x_3^{\mathrm{low}} - u_{\mathrm{sl}}(t) \le x_3(t) && t \in [0,T], \\
& u_{\mathrm{sl}}(t) \ge 0, && t \in [0,T].
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- Matlab implementation of iterative QP solver (`iQP`)
  - Uses parametric QP solver `qpOASES` [Ferreau et al 08;Potschka et al 10]
  - Tight tolerance ($\epsilon = 10^{-8}$)

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- Multiple shooting approach
  - Computation of full derivative matrices expensive
  - Here: Performance metric is number of matrix-vector products

Numerical Results NMPC

15 multiple shooting intervals

	QP Size		**Solve**	**Matrix-Vector Products**			
N_{PM}	n	m	Success	Min	Mean	Max	Full A_k
6	656	575	57	16	29.8	128	33
8	880	787	57	20	33.5	130	45
10	1,104	999	57	24	36.9	102	57
12	1,328	1,211	57	32	38.2	90	69
14	1,522	1,423	57	28	35.8	74	81

20 multiple shooting intervals

	QP Size		**Solve**	**Matrix-Vector Products**			
N_{PM}	n	m	Success	Min	Mean	Max	Full A_k
6	861	765	75	16	32.4	148	33
8	1,155	1,047	76	16	30.0	112	45
10	1,449	1,329	76	24	34.8	124	57
12	1,743	1,611	76	24	33.0	104	69
14	1,977	1,893	76	24	33.8	108	81

Numerical Results with Random NLPs

- Experiment with (exact) SQP method
  - QPs solved to tight tolerance (10^{-8} to 10^{-12})
- NLP with random perturbation (dense matrices)

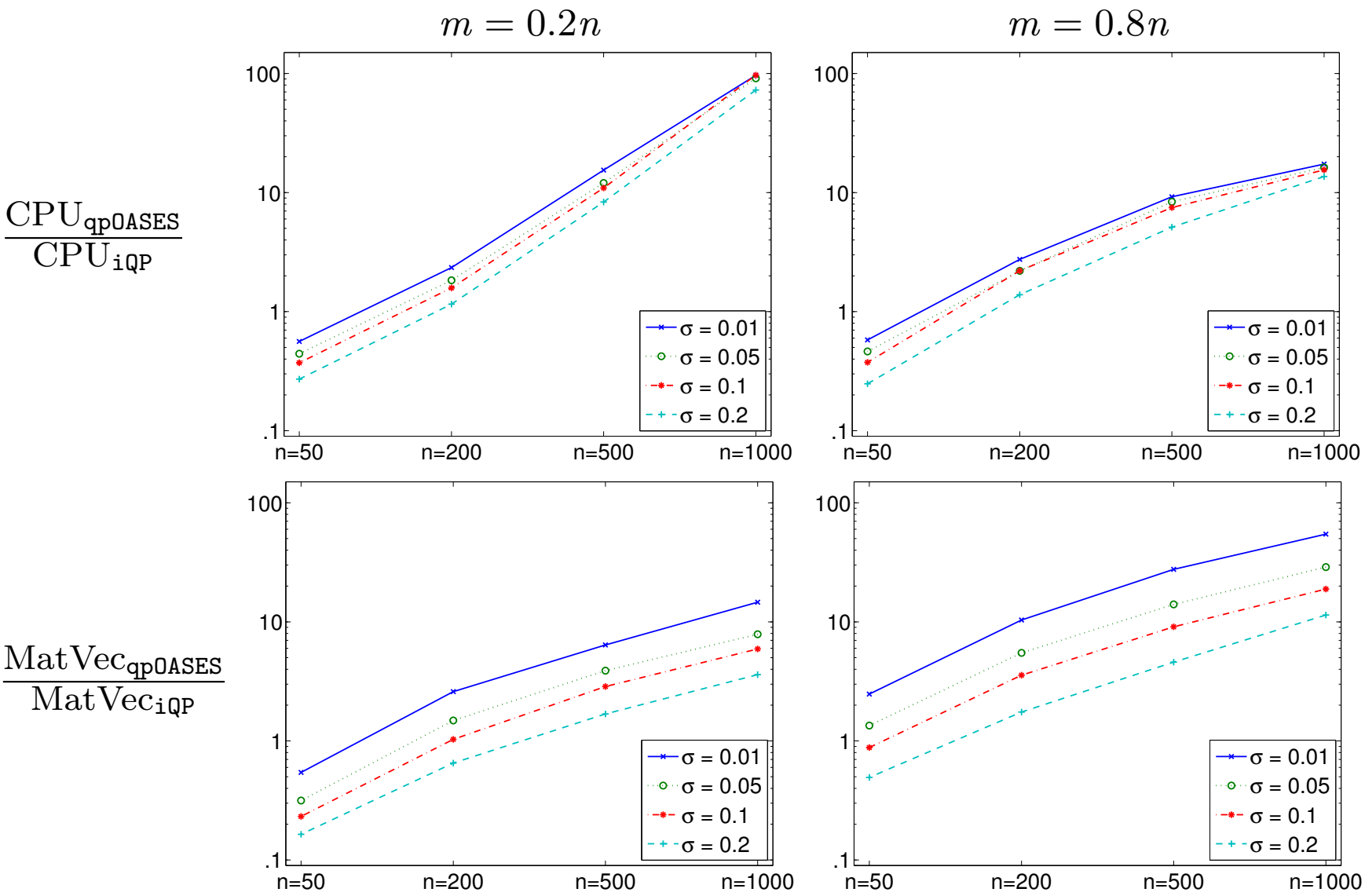
$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \tfrac{1}{2} x^T H_0 x + (q_0)^T x + r_0 \\ \text{s.t.} \quad & \tfrac{1}{2} x^T H_j x + (q_j)^T x + r_j \le 0 \qquad j = 1, \ldots, m \\ & x \ge 0 \end{aligned}$$

  - $n \in \{50, 200, 500, 1000\}$
  - $m \in \{0.2n, 0.5n, 0.8n, 1.5n\}$
  - $\sigma \in \{0.01, 0.05, 0.1, 0.2\}$ standard deviation of perturbation
- Solve sequences of 10 perturbed NLPs (3 for $n = 1000$)
- Solve first NLP, store final QP as reference QP

Hot-Started NLP Results

	Size of perturbation σ			
	0.01	0.05	0.1	0.2
Successfully solved	99.3%	99.3%	97.7%	42.9%
Avrg # `iQP` iters per SQP iter	4.9	8.5	12.5	20.6
Avrg change in # active ineq	5.3%	19.0%	25.7%	32.4%
Avrg change in # active bounds	3.4%	9.5%	13.8%	19.2%

Hot-Started NLP Results



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  - ...

THANK YOU!