Finite-time Switching Dynamics of Contact Mechanical Systems: A DVI and Hybrid System Perspective

Jinglai Shen

Department of Mathematics and Statistics
University of Maryland, Baltimore County, Baltimore, MD

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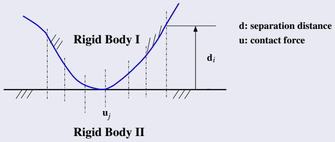
Outline

Unilateral Contact and Complementarity Formulation

Rigid-body Contact

Complementarity formulation:

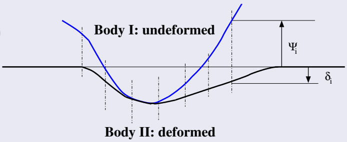
$$0 \leq u_i \perp d_i \geq 0, \forall\, i \;\Rightarrow 0 \leq u \perp d \geq 0$$



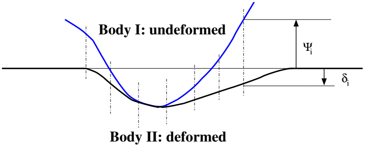
Compliant Contact

Complementarity formulation:

$$0 \leq u_i \perp \Psi_i + \delta_i \geq 0, \forall\, i \;\Rightarrow 0 \leq u \perp \Psi + \delta \geq 0$$



Differential Variational Inequality Formulation (I)



Compliant frictionless contact dynamics via differential complementarity system

$$
\begin{aligned}
M(q)\dot{v} &= D(q,v) + \frac{\partial \Psi^T}{\partial q}(q)u \\
\dot{q} &= G(q)v \\
0 \leq u &\perp \Psi(q) + \delta \geq 0
\end{aligned}
$$

where the body deformation δ satisfies:

$$
\text{Type I} : u = L(q, \delta), \quad \text{or} \quad \text{Type II} : u = K(q, \delta, \dot{\delta})
$$

Differential Variational Inequality Formulation (II)

Compliant frictional contact dynamics via DVI

$$
\begin{aligned}
M(q)\dot{v} &= D(q,v) + W(q)u \\
\dot{q} &= G(q)v \\
0 \le u_n &\perp s_n \ge 0 \\
(u_{i,t}, u_{i,o}) &\in \operatorname{argmin}_{(u^*_{i,t}, u^*_{i,o}) \in \mathcal{F}_i(\mu_i u_{i,n})} \left(u^*_{i,t} \cdot \dot{s}_{i,t} + u^*_{i,o} \cdot \dot{s}_{i,o} \right)
\end{aligned}
$$

where

- separation $s = \Psi(q) + \delta$
- local compliant force $u = K(q, \delta, \dot{\delta})$
- Coulomb friction cone $\mathcal{F}_i$ with friction coefficient μ_i

Hybrid System Formulation

Complementarity system

$$\dot{x} = F(x,u), \qquad 0 \leq u \perp H(x,u) \geq 0$$

Mode in terms of DAEs and mode switching

- Strong sense:

$$\dot{x} = F(x(t),u(t)),\ H_{\alpha(t)\cup\beta(t)}(x(t),u(t)) = 0,\ u_{\beta(t)\cup\gamma(t)}(t) = 0$$

where the fundamental indexes

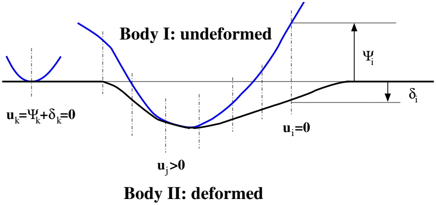
$$\begin{array}{lcll} \alpha(t) & = & \{\, i \, : \, u_i(t) > 0 = H_i(x(t),u(t)) \,\} & \text{the active } u\text{-indices} \\ \beta(t) & = & \{\, i \, : \, u_i(t) = 0 = H_i(x(t),u(t)) \,\} & \text{the degenerate indices} \\ \gamma(t) & = & \{\, i \, : \, u_i(t) = 0 < H_i(x(t),u(t)) \,\} & \text{the inactive } u\text{-indices} \end{array}$$

- Weak sense:

$$\dot{x}(t) = F(x(t),u(t)),\ H_\theta(x(t),u(t)) = 0,\ u_{\overline{\theta}}(t) = 0$$

where θ an index set and $\overline{\theta}$ is its complement.

Physical Interpretation



Modes in frictionless compliant contact dynamics

$$M(q)\dot{v} = D(q,v) + \frac{\partial \Psi^T}{\partial q}(q)u, \quad \dot{q} = G(q)v, \quad 0 \leq u \perp \Psi(q) + \delta \geq 0$$

(1) $u_i(t) = 0 < \left[\Psi(q(t)) + \delta(t)\right]_i \Rightarrow$ no contact;
(2) $u_j(t) > 0 = \left[\Psi(q(t)) + \delta(t)\right]_j \Rightarrow$ subject to contact;
(3) $u_k(t) = 0 = \left[\Psi(q(t)) + \delta(t)\right]_k \Rightarrow$ touch but without interaction

Finite-time Switching Dynamics

Switching dynamics

- Long-time switching dynamics: stability, positive invariance, ...
- Finite-time switching dynamics: (non-)Zeno behavior, simulation, ...

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Zeno behavior of switching systems

- Zeno: infinitely many mode switchings in finite time
- Non-Zeno: finite switchings in finite time

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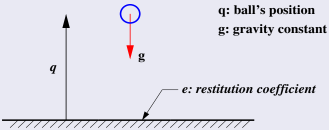
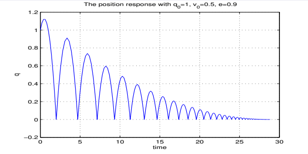
Example of Zeno contact mechanical system





Equations: $q > 0: \; \ddot{q} = -g; \quad q = 0: \; \dot{q}(t_+) = -e\,\dot{q}(t_-)$, where $e \in (0, 1)$.

Review of Zeno Analysis

Techniques for Zeno characterization

- geometric and dynamical system approach (Simic, Sastry, ...)
- Lyapunov approach for a Zeno equilibrium (Ames, Goebel, Or, Teel, ...)
- symmetry approach (Schumacher)
- others, especially in dynamical complementarity problems (DCPs) and queuing theory (Mendelbaum, Dai, Harrison, Bramson, ...)

Non-Zeno Switching Systems

Why non-Zeno dynamics?

- Scientific computing

  Impossible to simulate all mode switchings near a Zeno state.

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- Numerical analysis

  An optimal control problem with control constraints admits a second-order Runge-Kutta approximation, which relies on non-Zenoness of an optimal control solution.

  $\max_{1 \leq k \leq N_h} \|x_k^h - x^*(t_k)\| + \|\lambda_k^h - \lambda^*(t_k)\| + \|u_k^h - u^*(t_k + h/2)\| = O(h^2)$

A.L. Dontchev, W.W. Hager, and V.M. Veliov. Second-order Runge-Kutta approximations in control constrained optimal control. *SIAM Journal on Numerical Analysis*, Vol.38(1), pp. 202–226, 2000.

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- Systems and control analysis

  Many systems and control issues, e.g., observability and sensitivity analysis, depend on the non-Zeno property.

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Review of Non-Zeno Analysis

Challenges

1. Nonsmoothness: possibly non-unique solutions which are at best once (time-)differentiable
2. State dependent switching: implicit mode switchings at implicit switching times

Global approach

- Key idea: establish an upper bound on the number of switchings
- Examples:
  - Lipschitz piecewise linear systems: Pavol Brunovsky
  - Lipschitz piecewise analytic systems: Hector Sussmann

Solution expansion based local approach

- Key idea: show the absence of accumulation point (i.e., Zeno state)
- Examples:
  - Lipschitz complementarity systems: Pang, Camlibel, Han, Shen, et al
  - Well-posed non-Lipschitz bimodal piecewise linear/affine systems: Camlibel, Acary, Brogliato, et al

Non-Zeno Contact Systems (I)

Strongly regular complementarity system

Under (local) strong regularity of the underlying complementarity problem at (x^0, u^0), then the complementarity system is (locally) non-Zeno in the strong sense.

Application to frictionless compliant contact system

- Assumption: $\frac{\partial L(q,\delta)}{\partial \delta}|_{\delta=0}$ is positive definite for all q.
- for $(q(t_0), v(t_0))$ with $\|\delta(t_0)\|$ sufficiently small, $\exists\ \varepsilon > 0$ such that at each possible contact point, exactly one of the following three types occurs throughout $(t_0, t_0 + \varepsilon]$ and $[t_0 - \varepsilon, t_0)$: (1) no contact; (2) subject to contact; (3) touch but without interaction.

J.-S. Pang and J. Shen. Strongly regular differential variational systems. *IEEE Trans. Automatic Control*, Vol. 52, pp. 242–255, 2007.

Non-Zeno Contact Systems (II)

Differential quasi-variational inequality (DQVI)

$$\dot{x} = A(x,y) + B(x,y)u, \qquad 0 \leq y \perp G(x,y) \geq 0, \tag{1}$$

$$0 = C(x,y) + N(x)u - E^T\lambda, \quad 0 \leq \lambda \perp H(x,y) + Eu \geq 0 \tag{2}$$

Main result: if the first CP is (locally) strongly regular, N is (locally) P.D., and the second CP is solvable, then the DQVI is (locally) non-Zeno in the strong sense for the CP in (1) and in the weak sense for the CP in (2).

Application to frictional compliant contact system with polygonal friction cone

At each possible contact point, exactly one of (1)–(3) occurs in the normal direction, and at a strong contact point (i.e. (2)), $u_{i,t}$ either stays on or away from a side of the friction polygon throughout $(t_0, t_0+\varepsilon] \cup [t_0 - \varepsilon, t_0)$.

L. Han and J.-S. Pang. Non-Zenoness of a class of differential quasi-variational inequalities. *Math. Programming*, Series A, Vol. 121, pp. 171–1999, 2010.

Uncertainty analysis of Finite-time Switching Dynamics

Motivation

- parameter and/or initial state uncertainties
- modeling errors
- numerical errors
- perturbations

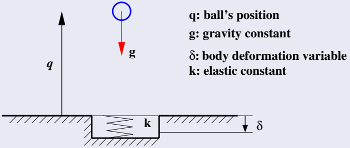
Robust Non-Zenoness: Motivating Example

Example: bouncing ball with elastic contact

The Lipschitz bimodal PA system:

$$m\ddot{q} = -mg + \max(0, -kq)$$

is non-Zeno. The unknown but bounded parameters m, k yield an infinite family of bimodal PA switching systems, denoted by $\mathcal{S}$.



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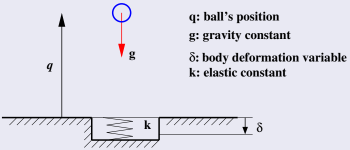

Non-Zenoness subject to uncertainties

- Is there a uniform bound on the number of switchings on a time interval $\mathcal{I}$ for all bimodal systems from $\mathcal{S}$, i.e., robust non-Zenoness?
- Can the robust non-Zenoness, if holds, be extended to a switching system with a discontinuous right-hand side?

Lipschitz Piecewise Affine System

Lipschitz PA system (PAS)

- Definition: $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz PA function
- Formulation based on polyhedral subdivision:

$$\dot{x} \;=\; A_i x + d_i, \quad \forall\, x \in \mathcal{P}_i = \{x \,|\, C_i x \geq b_i\}$$

  where $\{\mathcal{P}_i\}_{i=1}^m$ forms a polyhedral subdivision of $\mathbb{R}^n$.
- Well-posedness: the PAS has a unique C^1-trajectory $x(t, x_0)$ for all x_0.

Mode switching of PAS

- Definition: given $x(t, x_0)$, t_* is a non-switching time if $\exists$ a polyhedron $\mathcal{P}_i$ and $\varepsilon > 0$ such that $x(t, x_0) \in \mathcal{P}_i, \forall\, t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, t_* is a switching time, and there is a mode switching along $x(t, x_0)$ at t_*.
- PAS satisfies the simple switching property (Shen, Han, & Pang, 2010)
- Non-Zenoness: any $x(t, x_0)$ has finite many switchings in finite time (Camlibel, Pang, & Shen, SICON, 2006)

Robust Non-Zenoness: Main Result

Consider a family of Lipschitz PASs on $\mathbb{R}^n$:

$$\mathcal{S} = \{(A_{\sigma,i}, d_{\sigma,i}), \mathcal{P}_{\sigma,i}\}_{\sigma}, \text{ where } \sigma \text{ is the index.}$$

Standing assumptions

1. each PAS has at most m modes;
2. each polyhedron $\mathcal{P}_{\sigma,i}$ has at most p facets, i.e., $(n-1)$-dim faces.

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Main result

If $\sup_{\sigma,i}(\|A_{\sigma,i}\|) \leq \rho$ for some $\rho > 0$, then for any time interval $[0,T]$, there exists $N(\rho,T) \in \mathbb{N}$ such that there are at most $N(\rho,T)$ mode switchings on $[0,T]$ along any trajectory of each PAS from $\mathcal{S}$.

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Remarks

1. $N(\rho, T)$: independent of initial states and constant drift vectors $d_{\sigma,i}$
2. $N(\rho, T)$: only dependent on the uniform bound ρ of dynamic matrices $A_{\sigma,i}$ and the time length T

Robust Non-Zenoness: Discussions

Can the dynamic matrix bound be dropped?

- Answer: No!

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Example of the failure of robust non-Zenoness

A planar Lipschitz bimodal PLS on $\mathbb{R}^2$:

$$\dot{x} = Ax + b\max(0, -c^T x), \quad A = \begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \omega_1 > 0$$

- The eigenvalues of $A_1 = A$ and $A_2 = A - bc^T$ are $\pm\imath\,\omega_1$ and $\pm\,\imath\sqrt{\omega_1(\omega_1+1)}$ respectively
- the PLS has a periodic trajectory from any nonzero initial state with the constant period $\frac{\pi}{\omega_1} + \frac{\pi}{\sqrt{\omega_1(\omega_1+1)}}$ (via the Poincaré map)
- The number of mode switchings on $[0, T]$ along a nontrivial trajectory is roughly proportional to ω_1 for all ω_1 sufficiently large.
- $\|A_\sigma\| \to \infty \Longrightarrow \omega_{\sigma,1} \to \infty \Longrightarrow$ the number of switchings on $[0, T] \to \infty$

Robust Non-Zenoness: Sketch of the Proof (1)

Technical result: Sussmann's Lemma

Let $\kappa > 0$ and $n \in \mathbb{N}$. Let $\Delta T > 0$ be such that $\Delta T < \min\left(1, \frac{e^{-\kappa n}}{n^{\frac{3}{2}}\kappa}\right)$. If $\phi_1(t), \cdots, \phi_n(t)$ are absolutely continuous functions on a time interval I of length ΔT that satisfy a linear system of differential equations:

$$\dot{\phi}_i(t) = \sum_{j=1}^{n} \alpha_{ij}(t)\,\phi_j(t), \quad i = 1, \ldots, n, \quad \text{a.e. } I,$$

where the coefficients $\alpha_{ij}(t)$ are measurable real-valued functions on I such that $|\alpha_{ij}(t)| \leq \kappa$ for all $1 \leq i, j \leq n$ and all $t \in I$, then either (i) all the $\phi_i(t)$ vanish identically on I, or (ii) at least one of $\phi_i(t)$ has no zeros on I.

H.J. Sussmann. Bounds on the number of switchings for trajectories of piecewise analytic vector fields. *Journal of Differential Equations*, Vol. 43, pp. 399–418, 1982.

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Remarks

- it allows α_{ij} to vary for nonsmooth ϕ (independent of initial states)
- it gives an explicit upper bound on the number of switchings

H.J. Sussmann. Bounds on the number of switchings for trajectories of piecewise analytic vector fields. *Journal of Differential Equations*, Vol. 43, pp. 399–418, 1982.

Robust Non-Zenoness: Sketch of the Proof (2)

Major analytic difficulties

- nonsmoothness: $x(t)$ is once time differentiable
- complex algebraic relationships between dynamic matrices

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Stratgies

- nonsmoothness: apply combinatorial techniques
- algebraic relations: exploit the structure of polyhedral subdivisions

Robust non-Zenoness: Sketch of the Proof (3)

Bimodal PLS

Bimodal PLS: $\dot{x} = Ax + b\max(0, -c^T x)$; hyperplane: $h(x) = c^T x = 0$

Combinatorial technique

Given $x(t)$ and a time interval $\mathcal{I} := [0, T]$, if $h(x(t))$ has

- 2 zeros on $\mathcal{I}$: $\exists\, S_1 \in \{A, A - bc^T\}$ s.t. $c^T S_1 x(t)$ has a zero
- $3 \cdot 2$ zeros on $\mathcal{I}$: $\exists\, S_1, S_2 \in \{A, A - bc^T\}$ s.t. $c^T S_2 S_1 x(t)$ has a zero
- $\vdots \qquad \vdots \qquad \vdots$
- $\prod_{i=1}^{k}(2^{i-1}+1)$ zeros on $\mathcal{I}$: $\exists\, S_1, \ldots, S_k \in \{A, A - bc^T\}$ such that $c^T S_k \cdots S_1 x(t)$ has a zero
- $\vdots \qquad \vdots \qquad \vdots$
- $\prod_{i=1}^{n-1}(2^{i-1}+1)$ zeros on $\mathcal{I}$: $\exists\, \psi = \big(I, S_{11}, S_{21}S_{22}, \cdots, \prod_{j=1}^{n-1} S_{(n-1)j}\big)$ with $S_{ij} \in \{A, A - bc^T\}$ such that $q_1^{\psi}(t) = c^T x(t)$ and $q_k^{\psi}(t) = c^T \{\prod_{j=1}^{k-1} S_{(k-1)j}\} x(t), k = 2, \ldots, n$ has zeros

J. Shen. Robust Non-Zenoness of piecewise analytic systems with applications to complementarity systems. *Proc. of ACC*, pp. 148–153, Baltimore, MD, 2010.

Robust non-Zenoness: Sketch of the Proof (4)

Set up equation

- Let $q(t) := \big(c^T x(t), c^T Ax(t)), \ldots, cA^{(n-1)}x(t)\big) \Longrightarrow$
  $\dot{q} = \widetilde{A}q + \widetilde{b}\max(0, -q_1)$
- State transformation:

$$q^{\psi} = \begin{pmatrix} q_1^{\psi} \\ q_2^{\psi} \\ \vdots \\ q_n^{\psi} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ \star & 1 & & \\ & & \ddots & \\ \star & \cdots & \star & 1 \end{bmatrix}}_{P^{\psi}} q,$$

  where $\star$: either zero or a finite product of $c^T A^j b$ for $j = 0, 1, \ldots, n-1$.
- $|c^T A^j b|$ for $j = 0, 1, \ldots, n-1$: uniformly bounded.
- Equation:

$$\dot{q^{\psi}} = \widetilde{A}^{\psi} q^{\psi} + \widetilde{b}^{\psi} \max(0, -q_1^{\psi}),$$

  where $\widetilde{A}^{\psi} = P^{\psi}\widetilde{A}(P^{\psi})^{-1}$ and $\widetilde{b}^{\psi} = P^{\psi}\widetilde{b}$.

Robust non-Zenoness: Sketch of the Proof (5)

Uniform bound on system data

Let $\widetilde{c} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$ and define

$$\kappa(\rho) := \sup_{\sigma,\, \psi} \left(\, \|\widetilde{A}^{\psi}_{\sigma}\|, \, \|\widetilde{A}^{\psi}_{\sigma} - \widetilde{b}^{\psi}_{\sigma} \, \widetilde{c}^{T}\| \, \right).$$

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Uniform bound on the number of switchings

$$N(\rho, T) = \left(\frac{2T}{\min(1, e^{-\kappa n}/(n^{3/2}\kappa))} + 1 \right) \prod_{i=1}^{n-1} (2^{i-1} + 1)$$

Robust non-Zenoness: Sketch of the Proof (6)

Discussion for general PAS

1. Relation between dynamic matrices (relying on the geometry of polyhedral subdivision):

   Given $x^* \in \mathbb{R}^n$ and the index set $\mathcal{I}(x^*) := \{i : x^* \in \mathcal{P}_i\}$. For each $i \in \mathcal{I}(x^*)$, define $\mathcal{L}(x^*, i) := \{j : (C_i x^*)_j = 0\}$. For a fixed $i_* \in \mathcal{I}(x^*)$, there exist $b_{\ell,j} \in \mathbb{R}^n$ and scalars $\gamma_{\ell,j}$ such that for any $\ell \in \mathcal{I}(x^*)$,

   $$A_\ell = A_{i_*} + \sum_{j \in \mathcal{L}(x^*,i), i \in \mathcal{I}(x^*)} b_{\ell,j} (C_i)_{j\bullet}, \quad d_\ell = d_{i_*} - \sum_{j \in \mathcal{L}(x^*,i), i \in \mathcal{I}(x^*)} \gamma_{\ell,j} b_{\ell,j}.$$

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2. Goal of the proof: establish a uniform bound on the number of critical zero points using polyhedral subdivision and Sussmann's Lemma

Extension to Non-Lipschitz PLS

Non-Lipschitz piecewise linear system (PLS)

$$\dot{x} = A_i x, \qquad \forall\, x \in \mathcal{P}_i,$$

where $\{\mathcal{P}_i\}_{i=1}^m$ form a polyhedral subdivision of $\mathbb{R}^n$. The right-hand side need not be continuous on boundary of each $\mathcal{P}_i$.

Solution concepts

- Carathéodory solution
  - Forward Carathéodory solution at t_0 : a weak solution $x(t)$ s.t. $\exists$ the ith mode and $\varepsilon_{t_0} > 0$ s.t. $\dot{x}(t) = A_i x(t)$, $\forall\, t \in (t_0, t_0 + \varepsilon_{t_0})$.
  - This concept rules out the existence of a left Zeno time but allows a right Zeno time.
- Filippov solution
  - defined by the differential inclusion $\dot{x}(t) \in F(x(t))$ a.e.,

$$F(x) := \begin{cases} A_i x + d_i & \text{if } x \in \text{int}\mathcal{P}_i \\ \text{conv}(\{A_i x + d_i : i \in \mathcal{I}(x)\}) & \text{if } x \in \cap_{i \in \mathcal{I}(x)} \mathcal{P}_i \end{cases}$$

where $\mathcal{I}(x) := \{i \,:\, x \in \mathcal{P}_i\}$

Well-posedness of Non-Lipschitz PLS: Bimodal Case

Well-posedness (existence and uniqueness)

- Bimodal PLS on $\mathbb{R}^n$:

$$\dot{x} = \begin{cases} A_1 x & \text{if } c^T x \geq 0 \\ A_2 x & \text{if } c^T x \leq 0 \end{cases}$$

- Assumption: both (c^T, A_1) and (c^T, A_2) are observable pairs, i.e., $T_i := \begin{bmatrix} c & A_i^T c & \cdots & (A_i^{n-1})^T c \end{bmatrix}^T \in \mathbb{R}^{n\times n}, \quad i = 1, 2$ are of full rank.
- The bimodal PLS has a unique forward Carathéodory solution for any x_0 if and only if there is a lower triangular matrix $M = (m_{ij})$ with positive diagonal entries such that $T_1 = MT_2$.
- Non-Zenoness under the well-posedness condition (Camlibel, 2008): The bimodal PLS is non-Zeno if it is well posed (in the forward Carathéodory sense).

• D.E. Stewart. Uniqueness for solutions of differential complementarity problems. *Math. Programming*, Vol.118, pp. 327–345, 2009.

• J. Imura and A.J. Van der Schaft. Characterization of well-posedness of piecewise linear systems. *IEEE Trans. on Automatic Control*, Vol. 45(9), pp. 1600–1619, 2000.

Robust Non-Zenoness of Non-Lipschitz Bimodal PLS

Setting

Consider a family of well-posed bimodal PLSs $(A_{\sigma,1}, A_{\sigma,2}, c_\sigma, M_\sigma)$, where

- $T_{\sigma,i} = \begin{bmatrix} c_\sigma & A^T_{\sigma,i} c_\sigma & \cdots & (A^{n-1}_{\sigma,i})^T c_\sigma \end{bmatrix}^T$ has rank n, $i = 1, 2$
- $T_{\sigma,1} = M_\sigma T_{\sigma,2}$, where M_σ is a lower triangular matrix with positive diagonal entries
- Let
$$W_{\sigma,i} = T_{\sigma,i} A_{\sigma,i} T^{-1}_{\sigma,i}, \qquad i = 1, 2$$
- Fix σ. Let $q(t) := \big(c^T x(t), c^T A_1 x(t), \cdots, c^T A_1^{n-1} x(t)\big) = T_1 x(t)$. Then
$$\dot{q}(t) = \begin{cases} W_1 q(t) & \text{if } \ q_1(t) \geq 0 \\ M W_2 M^{-1} q(t) & \text{if } \ q_1(t) \leq 0 \end{cases}$$

Robust non-Zenoness

If there exist costants $0 < \nu < \mu$ such that $\nu \leq m_{\sigma,ii}/m_{\sigma,(i+1)(i+1)}$ for all σ and $\max_\sigma(\|W_{\sigma,1}\|, \|M_\sigma W_{\sigma,2} M_\sigma^{-1}\|) \leq \mu$, then the family of bimodal PLSs is robust non-Zeno (in the forward Carathéodory sense).

Application to Lipschitz Complementarity System

Linear complementarity system (LCS)

The LCS(A, B, C, D) on $\mathbb{R}^n$:

$$\dot{x} = Ax + Bu, \qquad 0 \leq u \perp Cx + Du \geq 0$$

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Modes and mode switching

- Fundamental index sets at each t for a given solution pair $(x(t), u(t))$

$$\begin{aligned} \alpha(t) &:= \{\, i \, : \, u_i(t) > 0 = (\, Cx(t) + Du(t)\,)_i \,\} \\ \beta(t) &:= \{\, i \, : \, u_i(t) = 0 = (\, Cx(t) + Du(t)\,)_i \,\} \\ \gamma(t) &:= \{\, i \, : \, u_i(t) = 0 < (\, Cx(t) + Du(t)\,)_i \,\} \end{aligned}$$

- A time t_* is a non-swithching time in the strong sense if there exist a scalar $\varepsilon > 0$ and $(\alpha_*, \beta_*, \gamma_*)$ such that

$$(\, \alpha(t), \beta(t), \gamma(t)\,) \quad = \quad (\alpha_*, \beta_*, \gamma_*\,), \quad \forall\, t \in (\, t_* - \varepsilon,\ t_* + \varepsilon\,);$$

  otherwise, t_* is a switching time in the strong sense.
- switching in the weak sense

Robust Non-Zenoness of Lipschitz LCS (I)

Main result

Let a family of LCS$(A_\sigma, B_\sigma, C_\sigma, D_\sigma)$, where σ is the index, be such that

- For each σ, SOL$(C_\sigma x, D_\sigma)$ is singleton for any $x \in \mathbb{R}^n$;
- $\{A_\sigma x + B_\sigma \text{SOL}(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz, piecewise linear functions, i.e., there exists $\kappa_1 > 0$ such that for each σ,

$$\big\|A_\sigma x + B_\sigma \text{SOL}(C_\sigma x, D_\sigma)\big\| \leq \kappa_1 \|x\|, \quad \forall\, x \in \mathbb{R}^n;$$

- There exists $\mu > 0$ such that $\|C_\sigma\| \leq \mu$ and $\|D_\sigma\| \leq \mu$ for all σ. Further, $\{\text{SOL}(C_\sigma x, D_\sigma)\}$ is a family of uniformly Lipschitz, piecewise linear functions, i.e., there exists $\kappa_2 > 0$ such that for each σ,

$$\big\|\text{SOL}(C_\sigma x, D_\sigma)\big\| \leq \kappa_2 \|x\|, \quad \forall\, x \in \mathbb{R}^n.$$

Then the family of the LCSs is robust strongly non-Zeno, i.e.,
given $[0, T]$, there exists $N(T, \kappa_i, \mu) \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^n$ and any index σ, $x_\sigma(t, x_0)$ has at most $N(\mathcal{I}, \kappa_i, \mu)$ switchings in the strong sense on $[0, T]$.

Robust Non-Zenoness of Lipschitz LCS (II)

Special case: LCS with the P-property

Consider a family of LCS$(A_\sigma, B_\sigma, C_\sigma, D_\sigma)$, where each D_σ is a P-matrix. If there exist $\kappa > 0$ and $\chi > 0$ such that

- $\max_\sigma(\|A_\sigma\|, \|B_\sigma\|, \|C_\sigma\|, \|D_\sigma\|) \leq \kappa$;
- $\varphi(D_\sigma) \geq \chi$ for all σ, where $\varphi(D)$ is the smallest principal minor of the matrix D.

Then the family of the LCSs is robust strongly non-Zeno.

Other cases in the weak sense

- Cone LCS:
$$\dot{x} = Ax + Bu, \quad \mathcal{C} \ni u \perp Cx + Dz \in \mathcal{C}^*,$$
where $\mathcal{C}$ is a polyhedral cone
- PSD-plus LCS:
$$\dot{x} = Ax + BF^T u, \quad 0 \leq u \perp FCx + FDF^T u \geq 0,$$
where D is a PD matrix

Future Research

Open Questions

1. robust non-Zenoness of general Lipschitz system, e.g., piecewise analytic systems
2. the connection between uniqueness and non-Zenoness of non-Lipschitz PASs
3. applications of robust non-Zenoness of Lipschitz PASs: given $[0, T]$, is it true that

$$\inf_{x^0, \sigma, i \in \{0, \ldots, N_{\sigma,T}(x^0)-1\}} \left| t_{\sigma,i+1}(x^0) - t_{\sigma,i}(x^0) \right| > 0,$$

   where $t_{\sigma,i}(x^0)$ is the i-th switching time in $[0, T]$ along $x_\sigma(t, x^0)$, and $\sup_{x^0, \sigma} N_{\sigma,T}(x^0) \leq N_T^*$.
4. potential applications to contact mechanical systems: sensitivity and uncertainty analysis

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Thank you!