Favorable Mathematical Properties of Frictional Contact Problems with Local Compliance

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Background and Setting

Background.

• As an alternative to a rigid-body model, a complementarity-based, 3-dimensional frictional contact model with local compliance and damping was introduced in the Ph.D. thesis of Peng Song in 2002.

Subsequently, we investigated the mathematical properties of the compliance model, putting them on a firm ground for further analysis, implementation, and extensions.

Applicable to

• both a dynamic model and a quasistatic model (inertia effects are ignored),

• a broad class of friction cones that include the quadratic Coulomb cone and its polygonal approximations;

• also to the "dual-cone modification" by Anitescu of the rigid-body model.

Caveat of the analysis: A special setting

• There is no damping in the normal contact forces but there is coupled stiffness between the normal and tangential forces via body deformations.

[Welcome discussion of the realism of the above assumption.]

Based on 3 papers:

L. Han and J.S. Pang.
Non-Zenoness of a class of differential quasi-variational inequalities. *Mathematical Programming, Series A* 121 (2010) 171-199.

J.S. Pang.
Frictional contact models with local compliance: Semismooth formulation. *Zeitschrift für Angewandte Mathematik und Mechanik* 88 (2008) 454–471.

J.S. Pang and D.E. Stewart.
Solution dependence on initial conditions in differential variational inequalities. *Mathematical Programming, Series B* 116 (2009) 429–460.

Main results

• Formulation as an ordinary differential equation (ODE) with a boundedly Lipschitz continuous, albeit implicitly defined, semismooth* right-hand side with global linear growth.

• Existence and uniqueness of a continuously differentiable solution trajectory originated from an arbitrary initial state.

• Finite (instead of impulsive) contact forces that are semismooth functions of the system state.

• Semismooth dependence of the trajectory on the initial state.

• Convergence of a shooting method for solving two-point boundary problems.

• In the case of a polygonal friction law, Zeno states are provably non-existent.

• **Open questions:**

— Allowing normal damping

— Zeno states with Coulomb friction.

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* Semismoothness is the next best thing to differentiability.

The Frictional Contact Model with Compliance

6 components

- Newton's law to describe force-induced body dynamics
- Kinematics to specify body orientation due to rotational motion
- **Constitutive law of compliance**
- Principle of non-penetration
- Friction law (maximum energy dissipation)
- Initial and/or boundary conditions.

Model Notations

Dimensions and constants

n_q, n_ν, n_δ	positive dimensional integers (for generalized coordinates, velocities, and contacts, respectively)
$\mu > 0$	n_δ-dimensional vector of friction coefficients at the contacts
q^0, ν^0, δ^0	initial states (generalized coordinates, velocities, and deformations, respectively) of the system
$T > 0$	time duration

Solution trajectories

q	n_q-vector of generalized coordinates of the bodies in contact
ν	n_ν-vector of generalized velocities of the bodies
$\lambda_{\mathrm{n;t;o}}$	3 n_δ-vectors of contact forces in the normal, labeled "n", and the two tangential directions, labeled "t" and "o", respectively
$\delta_{\mathrm{n;t;o}}$	3 n_δ-vectors of the body deformations in the normal and the two tangential directions
$s_{\mathrm{n;t;o}}$	3 n_δ-vectors of separations in the normal and two tangential directions

Model Notations (cont.)

Model functions (displacement dependent, yet time independent)

$M(q)$	$n_\nu \times n_\nu$ symmetric positive definite mass-inertia matrix
$f(q,\nu)$	n_ν-dimensional vector of external forces applied to the system
$G(q)$	$n_q \times n_\nu$-parametrization matrix describing system orientation
	$G(q)^T G(q)$ equals the identity matrix of order n_ν
$\Psi_{\mathrm{n;t;o}}(q)$	3 n_δ-dimensional distance and displacement functions
	in the normal and the two tangential directions
$K(q)$	$3n_\delta \times 3n_\delta$ symmetric positive definite system stiffness matrix
$C(q)$	$3n_\delta \times 3n_\delta$ symmetric positive (semi)definite system damping matrix
$W(q)$	$\triangleq \left[\ W_{\mathrm{n}}(q) \quad W_{\mathrm{t}}(q) \quad W_{\mathrm{o}}(q)\ \right]$ system wrench matrix
	$\triangleq G(q)^T \left[\ J_q\Psi_{\mathrm{n}}(q) \quad J_q\Psi_{\mathrm{t}}(q) \quad J_q\Psi_{\mathrm{o}}(q)\ \right] \in \mathbb{R}^{n_\nu \times 3n_\delta}$

all satisfying some technical assumptions such as uniformly positive definiteness, bounded Lipschitz continuity, and semismoothness.

The Model Conditions

- **Force equilibrium:** $M(q)\,\dfrac{d\nu}{dt} = f(q,\nu) + W(q)\lambda$ (ODE)

- **Kinematics:** $\dfrac{dq}{dt} = G(q)\nu$ (ODE)

- **Definition of separation:** $s = \delta + \Psi(q)$

- **Compliance law:** $\lambda = K(q)\delta + C(q)\dfrac{d\delta}{dt}$ (ODE)

- **Normal non-penetration and contact:** $0 \leq \lambda_{\mathrm{n}} \perp s_{\mathrm{n}} \geq 0$ (complementarity)

- **Tangential friction principle:** for each contact i, (cone minimization)

$$(\,\lambda_{i\mathrm{t}}, \lambda_{i\mathrm{o}}\,) \in \underset{(\,\widehat{\lambda}_{i\mathrm{t}}, \widehat{\lambda}_{i,\mathrm{o}}\,) \in \mathcal{F}_i(\mu_i \lambda_{i\mathrm{n}})}{\operatorname{argmin}} \left\{ \frac{ds_{i\mathrm{t}}}{dt}\,\widehat{\lambda}_{i\mathrm{t}} + \frac{ds_{i\mathrm{o}}}{dt}\,\widehat{\lambda}_{i\mathrm{o}} \right\}$$

- **Initial conditions:** $q(0) = q^0$, $\nu(0) = \nu^0$, and $\delta(0) = \delta^0$.

Friction Cones

described by set-valued maps $F : \mathbb{R}_+ \to \mathbb{R}^2$ with conic graphs;

prominent examples are:

- the Lorentz cone: $F^{\mathsf{C}}(\rho) \triangleq \left\{ (a,b) \in \mathbb{R}^2 \mid \sqrt{a^2+b^2} \leq \rho \right\}$ for $\rho \geq 0$

- its polygonal approximation:

$\widehat{F}(\rho) \triangleq \{ (a,b) \in \mathbb{R}^2 \mid a \cos\theta_j + b \sin\theta_j \leq \rho, \; j = 1, \cdots, \ell \}$,

for various angles $\theta_j \in [0^\circ, 360^\circ)$ such that the polygonal region $\widehat{F}(1)$ either inscribes or circumscribes the unit circle

- a special cone: $F^{\mathsf{P}}(\rho) \triangleq \{ (a,b) \in \mathbb{R}^2 \mid \max(|a|, |b|) \leq \rho \}$, corresponding to $\ell = 4$ and $\theta_j \in \{ 0^\circ; 90^\circ; 180^\circ; 270^\circ \}$.

No Normal Damping

The damping matrix:

$$C(q) \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{\mathsf{tt}}(q) & C_{\mathsf{to}}(q) \\ 0 & C_{\mathsf{ot}}(q) & C_{\mathsf{oo}}(q) \end{bmatrix}$$

with $\begin{bmatrix} C_{\mathsf{tt}}(q) & C_{\mathsf{to}}(q) \\ C_{\mathsf{ot}}(q) & C_{\mathsf{oo}}(q) \end{bmatrix}$ positive definite

The stiffness matrix:

$$K(q) \triangleq \begin{bmatrix} K_{\mathsf{nn}}(q) & K_{\mathsf{nt}}(q) & K_{\mathsf{no}}(q) \\ K_{\mathsf{tn}}(q) & K_{\mathsf{tt}}(q) & K_{\mathsf{to}}(q) \\ K_{\mathsf{on}}(q) & K_{\mathsf{ot}}(q) & K_{\mathsf{oo}}(q) \end{bmatrix}$$

Key steps in analysis

• The normal force λ_{n} is a bounded Lipschitz continuous, semismooth function of $(q, \delta_{\mathrm{t,o}})$.

• The tangential forces $\lambda_{\mathrm{t,o}}$ are uniquely determined, boundedly Lipschitz continuous, semismooth functions of the tuple $(q, \nu, \delta_{\mathrm{t,o}}, \lambda_{\mathrm{n}})$.

• Hence with $\mathbf{x} \triangleq (q, \nu, \delta_{\mathrm{t,o}})$, obtain the implicitly defined ODE: $\dot{\mathbf{x}}(t) = \Phi(\mathbf{x})$, where Φ is a boundedly Lipchitz continuous, semismooth function.

• Classical ODE theory is applicable to obtain existence and uniqueness; semismooth dependence on initial conditions requires advanced proof.

A variant. The linear compliance law $\lambda = K(q)\delta + C(q)\dot{\delta}$ can be replaced by a nonlinear law:

$$\begin{pmatrix} \lambda_{\mathrm{n}} \\ \lambda_{\mathrm{t}} \\ \lambda_{\mathrm{o}} \end{pmatrix} = \begin{pmatrix} L_{\mathrm{n}}(q, \delta_{\mathrm{n,t,o}}) \\ L_{\mathrm{t}}(q, \delta_{\mathrm{n,t,o}}, \dot{\delta}_{\mathrm{n,t,o}}) \\ L_{\mathrm{o}}(q, \delta_{\mathrm{n,t,o}}, \dot{\delta}_{\mathrm{n,t,o}}) \end{pmatrix},$$

provided some uniform strong monotonicity conditions are in place in such a law.

Analysis of Non-Zenoness: set-up

via a differential quasi-variational inequality formulation:

$$\begin{array}{rcll} \dot{x} & = & A(x,y) + B(x,y)u & \text{dynamics} \\ 0 \le y & \perp & G(x,y) \ge 0 & \text{normal non-penetration} \\ u & \in & \underbrace{\text{SOL}(K(x,y), C(x), N(x))}_{\text{solution set of an AVI}} & \text{tangential friction principle,} \end{array}$$

where $u \in \text{SOL}(K(x,y), C(x), N(x))$ means $u \in K(x,y)$ and

$$(\, v - v\,)^T \left[\, C(x,y) + N(x)u \,\right] \ge 0, \quad \text{for all } v \in K(x,y).$$

Excluding Coulomb cone: $K(x,y) \triangleq \left\{ u \in \mathbb{R}^\ell \mid H(x,y) + Eu \ge 0 \right\}$ is polyhedral.

Designation of variables in compliance contact model:

x state variables $(q, \nu, \delta_{\text{t,o}})$

y normal force λ_{n}

u tangential forces $\lambda_{\text{t,o}}$

What is a **Zeno state**, mathematically?

Algebraic Definition of a Zeno State

Associated with a solution trajectory $(x(t), y(t), u(t))$, define 5 index sets:

Complementarity (normal components):

$$\begin{array}{lcll} \alpha(t) & \triangleq & \{ i \mid : y_i(t) > 0 = G_i(x(t), y(t)) \} & \text{strongly active} \\ \beta(t) & \triangleq & \{ i \mid : y_i(t) = 0 = G_i(x(t), y(t)) \} & \text{degenerate} \\ \gamma(t) & \triangleq & \{ i \mid : y_i(t) = 0 < G_i(x(t), y(t)) \} & \text{strongly active} \end{array}$$

Variational condition (tangential components):

$$\begin{array}{lcl} \mathcal{I}(t) & \triangleq & \{ j \mid [\, H(x(t), y(t)) + Eu(t) \,]_i = 0 \} \\ \mathcal{J}(t) & \triangleq & \{ j \mid [\, H(x(t), y(t)) + Eu(t) \,]_i > 0 \}. \end{array}$$

Note: Activation/de-activation of constraint multipliers are not included in definition, due to possibility of non-unqueness of these multipliers that satisfy:

$$0 = C(x, y) + N(x)u + E^T \lambda$$

$$0 \leq \lambda \perp H(x, y) + Eu \geq 0.$$

Roughly, a state $(x(t_0), y(t_0), u(t_0))$ is non-Zeno if the above index sets stay constant in a small time interval to the left and right of the nominal time t_0.

Assumptions and Preliminary Consequences

Given a pair (x^0, y^0) satisfying $0 \leq y^0 \perp G(x^0, y^0) \geq 0$ (complementarity):

(A) the functions A, B, C, G, H, and N are analytic in a neighborhood of (x^0, y^0);

(B) y^0 is a strongly regular solution of the NCP: $0 \leq y \perp G(x^0, y) \geq 0$;

(C) the matrix $N(x^0)$ is positive definite (albeit not necessarily symmetric);

(D) $K(x, y) \neq \emptyset$ for all (x, y) in a neighborhood of (x^0, y^0) with $y \geq 0$.

It follows that

- open neighborhoods U and V of x^0 and y^0, respectively, and an analytic function $y : U \to V$ such that for each $x \in U$, $y(x)$ is the unique vector in V that solves the NCP: $0 \leq y \perp G(x, y) \geq 0$;

- for all (x, y) near (x^0, y^0) with $y \geq 0$, SOL$(K(x, y); C(x, y); N(x))$ is a singleton whose unique element is $u(x, y)$ is Lipschitz continuous and piecewise analytic in a neighborhood of (x^0, y^0).

The Main Non-Zeno Theorem

Let $t_0 > 0$ be a positive time (for forward and backward result). Under assumptions (A–D), there exist a scalar $\varepsilon > 0$ and ten index sets, $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ and $(\mathcal{I}_\pm, \mathcal{J}_\pm)$ such that

$$(\alpha(t), \beta(t), \gamma(t), \mathcal{I}(t), \mathcal{J}(t)) =$$

$$\begin{cases} (\alpha_-, \beta_-, \gamma_-, \mathcal{I}_-, \mathcal{J}_-) & \text{if } t \in [t_0 - \varepsilon, t_0) \\ (\alpha_+, \beta_+, \gamma_+, \mathcal{I}_+, \mathcal{J}_+) & \text{if } t \in (t_0, t_0 + \varepsilon, t_0]. \end{cases}$$

Moreover, the trajectory $(x(t), y(t), u(t))$ is analytic in the two open subintervals $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon, t_0)$. $\square$