Linear pencils encoded in the Newton polygon

(joint work with Wouter Castryck)

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Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

- k algebraically closed field of $\mathrm{char}(k) = 0$
- $\mathbb{T}^2 = (k^*)^2$ two-dimensional torus
- $f \in k[x^{\pm 1}, y^{\pm 1}]$ irreducible Laurent polynomial
- $U(f) \subset \mathbb{T}^2$ curve defined by f
- $\Delta = \Delta(f)$ the Newton polygon of f (i.e. the convex hull of the exponent vectors that appear in f with a non-zero coefficient)

Definition

f is non-degenerate with respect to its Newton polygon if for every face $\tau \subset \Delta(f)$ (including $\Delta(f)$ itself) the system

$$f_\tau = \frac{\partial f_\tau}{\partial x} = \frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$.

Definition

An algebraic curve C/k is called Δ-non-degenerate if it is birationally equivalent to $U(f)$ for some Δ-non-degenerate Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$.

Question

Which geometric properties/invariants of C are encoded in the combinatorics of the Newton polygon Δ?

Remark

Let Δ be a two-dimensional lattice polygon and consider the map

$$\varphi_\Delta : \mathbb{T}^2 \hookrightarrow \mathbb{P}^N : (x,y) \mapsto \left(x^i y^j\right)_{(i,j)\in\Delta\cap\mathbb{Z}^2} \quad \text{(where } N = \sharp(\Delta\cap\mathbb{Z}^2)-1\text{))}$$

Then $C = \overline{\varphi_\Delta(U(f))} \subset \mathrm{Tor}(\Delta) = \overline{\varphi_\Delta(\mathbb{T}^2)}$ is a hyperplane section.

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Definition

For lattice polygons $\Delta, \Delta' \subset \mathbb{R}^2$, we say that Δ is equivalent to Δ' (notation: $\Delta \cong \Delta'$) if Δ' is obtained from Δ through a unimodular transformation, i.e. through a transformation of the form

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} i \\ j \end{pmatrix} \mapsto A \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad A \in \mathrm{GL}_2(\mathbb{Z}),\ a_1, a_2 \in \mathbb{Z}.$$

Remark

If a Laurent polynomial

$$f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}(x,y)^{(i,j)}$$

is Δ-non-degenerate and α is a unimodular transformation, then

$$f^\alpha = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{i,j}(x,y)^{\alpha(i,j)}$$

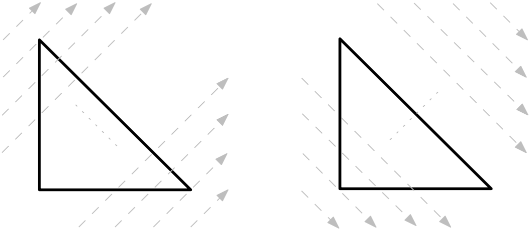
is $\alpha(\Delta)$-non-degenerate, and $U(f) \cong U(f^\alpha)$.

Definition

- A lattice direction is just a primitive element of $v = (a, b) \in \mathbb{Z}^2$.
- For a non-empty lattice polygon Δ and a lattice direction $v = (a, b)$, the width of Δ with respect to v is the minimal d for which there exists an $m \in \mathbb{Z}$ such that Δ is contained in the strip
$$m \leq aY - bX \leq m + d.$$
Note that $w(\Delta, v) = w(\Delta, -v)$.
- The lattice width of Δ is
$$\operatorname{lw}(\Delta) = \min_{v} w(\Delta, v).$$

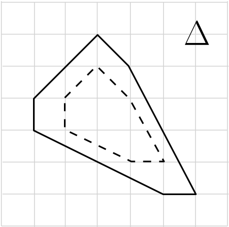
Example

The width of $d\Sigma$ with respect to $(1,1)$ is $2d$, while its width with respect to $(1,-1)$ is d. Here, $\Sigma = \mathrm{conv}\{(0,0),(1,0),(0,1)\}$.



The polygon $d\Sigma$ has lattice width $\mathrm{lw}(d\Sigma) = d$ and there are precisely three lattice directions computing this.

Example



Also in this case, there are three lattice width directions.

Definition

Let Δ, f, $C \subset \text{Tor}(\Delta)$, $v = (a, b)$ be as before.
Then the rational map $U(f) \to \mathbb{T}^1 : (x, y) \mapsto x^a y^b$ extends to a degree $w(\Delta, v)$ morphism $C \to \mathbb{P}^1$. Let g_v be the corresponding base-point free pencil. A pencil on C that arises as g_v for some lattice direction v is called combinatorial.

Remark

- Note that $g_v = g_{-v}$.
- The correspondence between pairs $\pm v$ of lattice directions and combinatorial pencils is usually 1-to-1, but there are counterexamples.

Definition

Let Δ be a lattice polygon and $v = (a, b)$ a lattice direction, such that $d = w(\Delta, v) \geq 2$. Assume that Δ is contained in the strip $m \leq aY - bX \leq m + d$. Then we define the width invariants of Δ with respect to v as the multiset

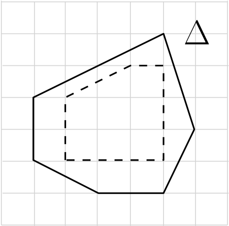
$$W(\Delta, v) = \{W_\ell\}_{\ell=1,\ldots,d-1}$$

where

$$W_\ell = \sharp \left\{ (i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2 \,\middle|\, aj - bi = m + \ell \right\} - 1.$$

Here, $\Delta^{(1)}$ is the convex hull of the interior lattice points of Δ.

Example



Here, $W(\Delta,(1,0)) = \{1,3,3,3\}$ and $W(\Delta,(0,1)) = \{2,2,3,3\}$.

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Theorem (H. Baker 1893, Khovanskii 1977)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the (geometric) genus g of $U(f)$ equals $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2)$.
Moreover, if $U(f)$ is non-hyperelliptic, the map

$$\varphi_{\Delta^{(1)}} : U(f) \to \mathbb{P}^{g-1} : (x, y) \mapsto \left(x^i y^j\right)_{(i,j) \in \Delta^{(1)} \cap \mathbb{Z}^2}$$

gives rise to a canonical model

$$C_{can} = \overline{\varphi_{\Delta^{(1)}}(U(f))} \subset Tor(\Delta^{(1)}) \subset \mathbb{P}^{g-1}.$$

Remark

The inclusion $C_{can} \subset \mathrm{Tor}(\Delta^{(1)})$ is not a hyperplane section!

Theorem (C.-C.)

Given a lattice polygon Δ with $\Delta^{(1)}$ two-dimensional and f non-degenerate w.r.t. Δ, there is a concrete way to write down generators of the canonical ideal $\mathcal{I}(C_{can})$.

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

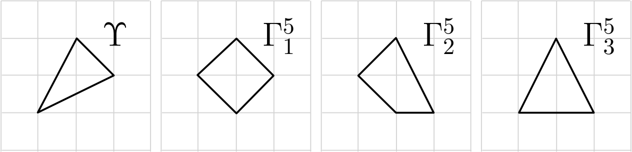
Two applications and more questions

Theorem (R. Kawaguchi, C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Suppose that $\Delta^{(1)}$ is not equivalent to any of the following:

$$\emptyset, \quad (d-3)\Sigma \text{ (for some integer } d \geq 3), \quad \Upsilon, \quad 2\Upsilon, \quad \Gamma^5_1, \quad \Gamma^5_2, \quad \Gamma^5_3.$$

Then every gonality pencil on (the smooth projective model of) $U(f)$ is combinatorial.



Remark

- If $\Delta^{(1)} = \emptyset$ then $U(f)$ is rational, hence of gonality 1.
- If $\Delta^{(1)} \cong (d-3)\Sigma$ then $U(f)$ is birationally equivalent to a smooth projective plane curve of degree d, hence of gonality $d-1$.
- If $\Delta^{(1)} \cong \Upsilon$ then $U(f)$ is a non-hyperelliptic curve of genus 4, hence of gonality 3.
- If $\Delta^{(1)} \cong 2\Upsilon$ then $U(f)$ is birationally equivalent to a smooth intersection of two cubics in $\mathbb{P}^3$, hence of gonality 6.
- If $\Delta^{(1)} \cong \Gamma_i^5$ $(i = 1, 2, 3)$ then $U(f)$ is a non-hyperelliptic, non-trigonal curve of genus 5, hence of gonality 4.

Corollary

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate. Then the gonality $\gamma(U(f))$ of $U(f)$ equals $lw(\Delta^{(1)}) + 2$, unless $\Delta^{(1)} \cong \Upsilon$ (i.e. $\Delta \cong 2\Upsilon$), in which case it equals 3.

Corollary

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate.

- *If $\Delta^{(1)} = \emptyset$ then there is a unique gonality pencil.*
- *If $\Delta^{(1)} \cong \Upsilon$ then the number of gonality pencils is at most 2.*
- *If $\Delta^{(1)} \cong (d-3)\Sigma$ for some $d \geq 3$, or if $\Delta^{(1)} \cong 2\Upsilon, \Gamma_1^5, \Gamma_2^5, \Gamma_3^5$, then there are infinitely many gonality pencils.*
- *In all other cases the number of gonality pencils equals the number of lattice width directions. In particular, the number of gonality pencils is at most 4, and the bound is met iff $\Delta^{(1)} \cong d\Gamma_1^5$ for some $d \geq 2$.*

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Definition

If $\Delta \neq \emptyset$, then its lattice size $\mathrm{ls}(\Delta)$ is defined as the minimal integer $d \geq 0$ such that Δ is equivalent to a lattice polygon that is contained in $d\Sigma$ (set $\mathrm{ls}(\emptyset) = -2$).

Theorem (C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$. Then the minimal degree of a (possibly singular) projective plane curve that is birationally equivalent to $U(f)$ is bounded by $\mathrm{ls}(\Delta^{(1)}) + 3$. If $\Delta^{(1)} \cong (d-1)\Upsilon$ for a certain integer $d \geq 2$ (i.e. $\Delta \cong d\Upsilon$), then it is moreover bounded by $3d - 1$.

Definition

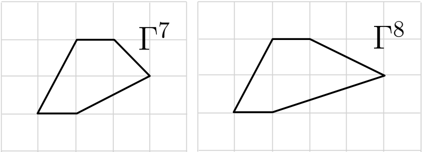
By a near-gonal pencil on a smooth projective curve C/k, we mean a base-point free $g^1_{\gamma(C)+1}$ (note that such pencils need not exist).

Theorem (C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be Δ-non-degenerate and let γ be the gonality of $U(f)$. Suppose that

$$ls(\Delta^{(1)}) \geq lw(\Delta^{(1)}) + 2 \qquad (\bigstar)$$

and that $\Delta^{(1)} \not\cong 2\Upsilon, 3\Upsilon, \Gamma^7, \Gamma^8$. Then every base-point free $g^1_{\gamma+1}$ on (the smooth projective model of) $U(f)$ is combinatorial.



Remark

- If condition ($\bigstar$) fails, then $U(f)$ is birationally equivalent with a (possibly singular) plane curve of degree $\gamma + 1$ or $\gamma + 2$, so $U(f)$ has infinitely many base-point free $g^1_{\gamma+1}$'s.
  This is also the case if $\Delta^{(1)} \cong 2\Upsilon$ (with $\gamma = 6$) or $\Delta^{(1)} \cong \Gamma^7$ (with $\gamma = 4$).
- If $\Delta^{(1)} \cong 2\Upsilon$, then there exists a base-point free g^1_5, but no combinatorial one.
- If $\Delta^{(1)} \cong \Gamma^8$, there are no combinatorial g^1_5's, but there are instances of curves $U(f)$ with a base-point free g^1_5.

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Definition

To a smooth projective curve C/k (of genus $g \geq 4$), one can associate its Clifford index

$$\text{ci}(C) = \min\{\, d - 2r \mid C \text{ carries a special divisor } D \text{ with } |D| = g^r_d \}$$

and its Clifford dimension

$$\text{cd}(C) = \min\{\, r \mid \text{there exists a } g^r_d \text{ realizing } \text{ci}(C)\}.$$

Note that D is special iff $h^0(C, D), h^0(C, K_C - D) \geq 2$.

Theorem (R. Kawaguchi, C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$ and suppose that $\sharp(\Delta^{(1)} \cap \mathbb{Z}^2) \geq 4$. Then

- *if $\Delta^{(1)} \cong (d-3)\Sigma$ for $d \geq 5$, then $ci(U(f)) = d - 4$ and $cd(U(f)) = 2$,*
- *if $\Delta^{(1)} \cong \Upsilon$, then $ci(U(f)) = 1$ and $cd(U(f)) = 1$,*
- *if $\Delta^{(1)} \cong 2\Upsilon$, then $ci(U(f)) = 3$ and $cd(U(f)) = 3$,*
- *in all other cases, $ci(U(f)) = lw(\Delta^{(1)})$ and $cd(U(f)) = 1$.*

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Definition

Consider linear subspaces $\mathbb{P}^{e_1}, \ldots, \mathbb{P}^{e_n} \subset \mathbb{P}^N$ that span $\mathbb{P}^N$ (with $0 \leq e_1 \leq \ldots \leq e_n$). In each $\mathbb{P}^{e_\ell}$, take a rational normal curve of degree e_ℓ, e.g. parameterized by

$$\nu_\ell : \mathbb{P}^1 \to \mathbb{P}^{e_\ell} : (X : Y) \mapsto \left(X^{e_\ell} : X^{e_\ell - 1}Y : \cdots : Y^{e_\ell}\right).$$

Then

$$S = \bigcup_{P \in \mathbb{P}^1} \langle \nu_1(P), \ldots, \nu_n(P) \rangle \subset \mathbb{P}^N$$

is a rational normal scroll of type $(e_1, \ldots, e_n)$.

Remark

- the degree of S is $e_1 + \ldots + e_n$, so S has minimal degree.
- S is smooth iff $e_1 > 0$.

Definition

Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of genus $g \geq 3$ and fix any pencil g^1_d on C. By Riemann-Roch theorem, the dimension of the linear span $\langle D \rangle$ equals $d - h^0(C, D)$ for any $D \in g^1_d$. Consider

$$S = \bigcup_{D \in g^1_d} \langle D \rangle \subset \mathbb{P}^{g-1}.$$

It is clear that S is a n-dimensional scroll containing the curve C, with $n = d - h^0(C, D) + 1$. One can show that S is actually a rational normal scroll of a certain type $(e_1, \ldots, e_n)$ and we call the numbers $e_1, \ldots, e_n$ the scrollar invariants of C w.r.t. g^1_d.

Remark

If $g^1_d = |D|$ is complete and base-point free, then $n = d - 1$ and the scrollar invariants can alternatively be described as follows:

$$h^0(C, mD) = \begin{cases} h^0(C,(m-1)D) + 1 & \text{if } 0 \leq m \leq e_1 + 1, \\ h^0(C,(m-1)D) + 2 & \text{if } e_1 + 1 < m \leq e_2 + 1, \\ \vdots & \vdots \\ h^0(C,(m-1)D) + d - 1 & \text{if } e_{d-2} + 1 < m \leq e_{d-1} + 1, \\ h^0(C,(m-1)D) + d & \text{if } m > e_{d-1} + 1. \end{cases}$$

Theorem (C.-C.)

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, and assume that $\Delta^{(1)}$ is two-dimensional. Let v be a lattice direction. Then the multiset of scrollar invariants of $U(f)$ with respect to g_v equals the multiset of non-negative width invariants of Δ with respect to v.

Corollary

Let $f \in k[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta = \Delta(f)$, where we assume that $\Delta^{(1)}$ is two-dimensional. Let v be a lattice direction. Then g_v is complete if and only if the width invariants of Δ with respect to v are all non-negative.

Introduction

Definitions

Genus and canonical ideal

Gonality pencils

Near-gonality pencils

Clifford index and dimension

Scrollar invariants

Two applications and more questions

Theorem (C.-C.)

If a smooth projective curve C/k carries a point P having a Weierstrass semi-group of embedding dimension 2 (i.e. of the form $a\mathbb{N} + b\mathbb{N}$ for coprime integers $a, b \geq 2$), then this semi-group does not depend on the choice of P.

Theorem (C.-C.)

If a smooth projective curve C/k of gonality $\gamma > 2$ is contained in a Hirzebruch surface $\mathcal{H}_n$, then n is an invariant of C.

- Is the Newton polytope intrinsic for non-degenerate curves? More precise formulation: If a curve C/k is both Δ-non-degenerate and Δ'-non-degenerate, does it follow that $\Delta^{(1)} \cong \Delta'^{(1)}$?
- Is it possible to prove (or disprove) Green's canonical conjecture for non-degenerate curves?

Thanks!