Rank of divisors on curves and graphs under specialization: the hyperelliptic case and the genus 3 case
(joint work with Kazuhiko Yamaki)

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Plan of the talk

We study, in some sense, an equality condition of the specialization lemma.

1. Specialization lemma (Baker, Amini-Caporaso)
2. A question
3. Hyperelliptic case and genus 3 case (with an outline of proof)
4. Relation to the algebraic rank (introduced by Caporaso)

References

- *Ranks of divisor classes on hyperelliptic curves and graphs under specialization*, preprint arXiv:1304.6979, to appear in IMRN.
- *Algebraic rank on hyperelliptic graphs and graphs of genus* 3, preprint arXiv:1401.3935.

Part 1 Specialization lemma (Baker, Amini-Caporaso)

Setup

R : a complete discrete valuation ring (cDVR)

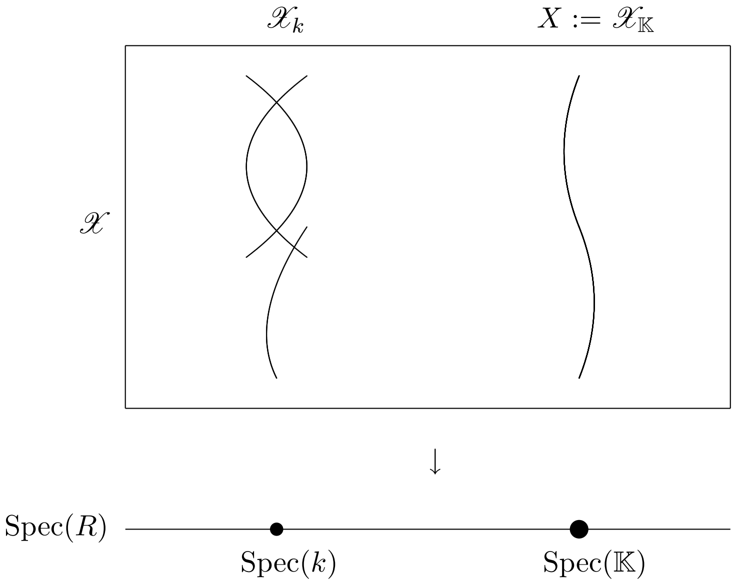
$\mathbb{K}$: fractional field of R, $\bar{\mathbb{K}}$: algebraic closure of $\mathbb{K}$

k : residue field of R, which we assume is algebraically closed

$\mathscr{X}$: regular, generically smooth, semi-stable R-curve
(projective and flat over R, $\mathscr{X}_k$ is a reduced nodal curve)

X : generic fiber of $\mathscr{X}$, i.e., $X = \mathscr{X}_{\mathbb{K}}$

$\mathscr{X}_k$: special fiber of $\mathscr{X}$


$\mathscr{X}_k$
$X := \mathscr{X}_{\mathbb{K}}$
$\mathscr{X}$
$\downarrow$
$\mathrm{Spec}(R)$
$\mathrm{Spec}(k)$
$\mathrm{Spec}(\mathbb{K})$

Reduction graph of $\mathscr{X}$

$\mathscr{X}_k$: the special fiber of $\mathscr{X}$,

a reduced curve with only nodes as singularities

$\bar{G} = (G, w)$: the (vertex-weighted) dual graph of $\mathscr{X}_k$

irreducible component C_v of $\mathscr{X}_k$ $\longleftrightarrow$ vertex v of G

point in $C_v \cap C_{v'}$ $\longleftrightarrow$ edge of G connecting v and v'

node in C_v $\longleftrightarrow$ loop edge at v

$V(G)$: the set of vertices of G

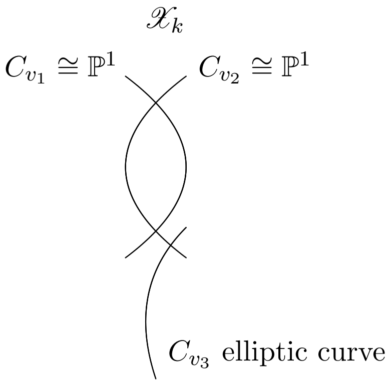
$E(G)$: the set of edges of G

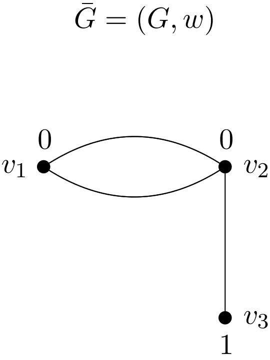
$w : V(G) \to \mathbb{Z}_{\geq 0}$, $\quad w(v) :=$ "geometric genus of C_v"

$\bar{\Gamma} = (\Gamma, w)$: the metric graph associated to $\bar{G}$,

where each edge of G is assigned length 1

$\Gamma_{\mathbb{Q}} := \{x \in \Gamma \mid \mathrm{dist}(x, v) \in \mathbb{Q} \text{ for } {}^{\forall} v \in V(G)\}$

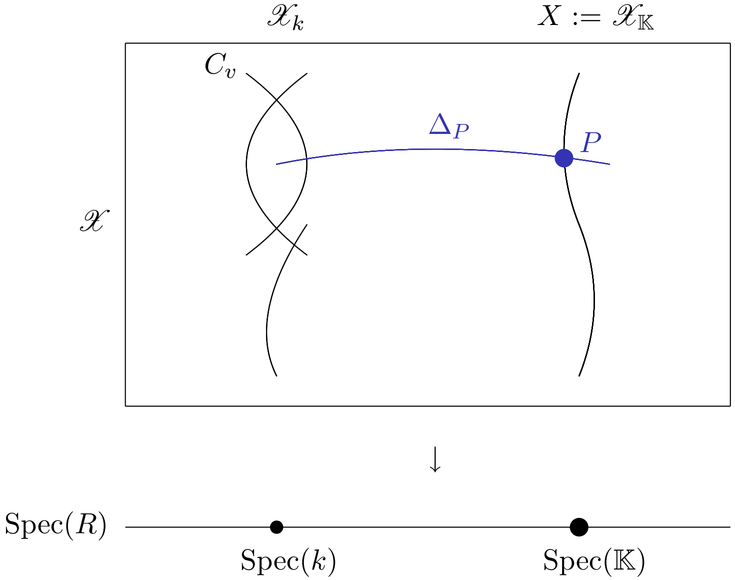




Specialization map

$P \in X(\mathbb{K})$: $\mathbb{K}$-valued point in the generic fiber X

Δ_P: Zariski closure of P in $\mathscr{X}$, which meets exactly one irreducible component of the special fiber $\mathscr{X}_k$ (by the valuative criterion of properness)


$\mathscr{X}_k$
$X := \mathscr{X}_{\mathbb{K}}$
C_v
Δ_P
P
$\mathscr{X}$
$\downarrow$
$\mathrm{Spec}(R)$
$\mathrm{Spec}(k)$
$\mathrm{Spec}(\mathbb{K})$

The assignment

$$\tau : X(\mathbb{K}) \to V(G), \quad P \mapsto v$$

gives the specialization map, which extends to

$$\tau_* : \mathrm{Div}(X_{\bar{\mathbb{K}}}) \to \mathrm{Div}(\Gamma_{\mathbb{Q}}) := \bigoplus_{x \in \Gamma_{\mathbb{Q}}} \mathbb{Z}[x].$$

Theorem (Specialization Lemma, Baker, Amini-Caporaso)

For any $\widetilde{D} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$*, put* $D := \tau_*(\widetilde{D}) \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$*. Then*

$$r_{\bar{\Gamma}}(D) \geq r_X(\widetilde{D}).$$

(Here, let $\Gamma^\bullet$ be the metric graph obtained by adding $w(v)$ loops (of any length) at v to Γ. Then $r_{\bar{\Gamma}}(D) = r_{\Gamma^\bullet}(D)$.)

Part 2 A question

R: a complete discrete valuation ring (cDVR)
with $\mathrm{Frac}(R) = \mathbb{K}$ and algebraically closed residue field k

$\bar{G} = (G, w)$: a vertex-weighted graph

$\bar{\Gamma} = (\Gamma, w)$: associated metric graph of $\bar{G}$

Question

Does there exist a regular, generically smooth, semi-stable R-curve $\mathscr{X}$ with reduction graph $\bar{G}$ satisfying the following condition (C)?

$^{\forall} D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, $^{\exists} \widetilde{D} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$ with $D = \tau_*(\widetilde{D})$ s.t.

$$r_{\bar{\Gamma}}(D) = r_X(\widetilde{D}).$$

R: a cDVR, $\bar{G} = (G, w)$: a vertex-weighted graph as before.

Observation

- If such $\mathscr{X}$ exists, then the Riemann–Roch formula for $\bar{G}$ is deduced from the Riemann–Roch formula for $X (:= \mathscr{X}_{\mathbb{K}})$. (This is our original motivation to consider the question.)
- If such $\mathscr{X}$ exists, then

$$\tau_* \left(W_d^r(X_{\mathbb{K}}) \right) = W_d^r(\bar{\Gamma}_{\mathbb{Q}})$$

  where $W_d^r(X_{\mathbb{K}}) := \{ \widetilde{D} \in \mathrm{Pic}(X_{\mathbb{K}}) \mid \deg(\widetilde{D}) = d,\ r_X(\widetilde{D}) \geq r \}$, and $W_d^r(\bar{\Gamma}_{\mathbb{Q}}) = \{ D \in \mathrm{Pic}(\Gamma_{\mathbb{Q}}) \mid \deg(D) = d,\ r_{\bar{\Gamma}}(D) \geq r \}$.
- The question is related to the algebraic rank (cf. Len's talk).

Observation (continued)

- From Jensen's talk on Monday (if I understand correctly):

  Γ: a generic rational metric graph of g-loops

  $\mathscr{X}$: any regular, generically smooth, strongly semi-stable R-curve with reduction graph Γ

  Then Cartwright–Jensen–Payne show

  ${}^{\forall}D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, ${}^{\exists}\widetilde{D} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$ with $D = \tau_*(\widetilde{D})$ s.t.

  $$r_\Gamma(D) = r_X(\widetilde{D}).$$

- From Cartwright's talk on Monday (if I understand correctly): Given $\bar{\Gamma}$ and an effective divisor $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, Cartwright studies if there exist $\mathscr{X}$ and an effective divisor $\widetilde{D} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$ with $D = \tau_*(\widetilde{D})$ and $r_\Gamma(D) = r_X(\widetilde{D})$.

Part 3 Hyperelliptic case and genus 3 case

$\bar{G} = (G, w)$: a vertex-weighted graph

$\bar{\Gamma} = (\Gamma, w)$: associated metric graph of $\bar{G}$

Definition (genus)

$g(G) = g(\Gamma) := |E(G)| - |V(G)| + 1$

$g(\bar{G}) = g(\bar{\Gamma}) := g(G) + \sum_{v \in V(G)} w(v)$

Definition (hyperelliptic vertex-weighted graph)

$\bar{G}$ is hyperelliptic $\iff$

$^{\exists} D \in \mathrm{Div}(G)$ s.t. $\deg(D) = 2$ and $r_{\bar{\Gamma}}(D) = 1$.

Definition (positive-type bridge)

$\bar{G} = (G, w)$: a vertex-weighted graph

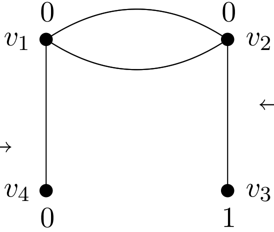
An edge e of G is a bridge $\iff G \setminus \{e\} = {}^{\exists}G_1 \amalg {}^{\exists}G_2$

A bridge e is of positive type $\iff g(\bar{G}_1) \geq 1$ and $g(\bar{G}_2) \geq 1$,

where $\bar{G}_i = (G_i, w|_{G_i})$.

Example

not a positive-type bridge →



← a positive-type bridge

Main results

Theorem (hyperelliptic case)

R: cDVR with algebraically closed residue field k

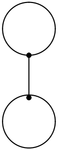
$\bar{G} = (G, w)$: *a hyperelliptic vertex-weighted graph*

Assume that $\mathrm{ch}(k) \neq 2$. *Then the following are equivalent.*

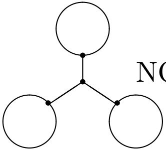
(i) *There exists a regular, generically smooth, semi-stable R-curve* $\mathscr{X}$ *with reduction graph* $\bar{G}$ *satisfying the condition (C) (i.e. the answer to the question is YES for* $\bar{G}$*).*

(ii) *For every vertex* v *of* G, *there are at most* $(2w(v) + 2)$ *positive-type bridges emanating from* v.

Example



YES



NO

Theorem (genus 3 case)

R: *cDVR with algebraically closed residue field k*

$\bar{G} = (G, w)$: *a vertex-weighted non-hyperelliptic graph of genus* 3

$\mathscr{X}$: *any regular, generically smooth, semi-stable R-curve* $\mathscr{X}$ *with reduction graph* $\bar{G}$

Then $\mathscr{X}$ *satisfies the condition (C)*

(in particular, the answer to the question is YES for $\bar{G}$*).*

Remark

For any vertex-weighted graph $\bar{G}$ of genus 0 or 1, the answer to the question is YES for $\bar{G}$. (Note that genus 2 graphs are hyperelliptic.)

Outline of proof of Theorem

We concentrate on the hyperelliptic case.

1. Construction of $\mathscr{X}$ (Caporaso, Amini–Baker–Brugallé–Rabinoff, K-Yamaki)
2. Computation of rank of reduced divisors on a hyperelliptic graph
3. Lifting of divisors

Remark

- In the above proof, we use Baker's specialization lemma. Strictly speaking, we don't use Baker–Norine's Riemann–Roch formula, but we do use very much the notion of v-reduced divisors.
- The proof of Theorem (genus 3 case), we use Amini–Caporaso's specialization lemma and Baker–Norine's Riemann–Roch formula.

Step 0: Some properties of hyperelliptic graphs

Hyperelliptic graphs have been studied by Baker–Norine, Chan, Caporaso ...

- $\bar{G} = (G, w)$: a vertex-weighted graph with $g(\bar{G}) \geq 2$
  Assume that any vertex $v \in V(G)$ of valence 1 satisfies $w(v) > 0$.
  Γ: associated metric graph of G
  $\Gamma^{\bullet}$: metric graph obtained by adding $w(v)$ loops at ${}^{\forall}v$ to Γ.
  Then the following are equivalent:
  (i) $\bar{G}$ is hyperelliptic;
  (ii) there exists a unique involution ι on $\Gamma^{\bullet}$ such that $\Gamma^{\bullet}/\iota$ is a tree.
- $\bar{G} = (G, w)$: a hyperelliptic vertex-weighted graph
  $\bar{\Gamma} = (\Gamma, w)$: associated metric graph of $\bar{G}$
  Γ': the metric graph that is obtained by contracting all edges with endpoint $v \in V(G)$ of valence 1 and $w(v) = 0$. We say $v_0 \in \Gamma'_{\mathbb{Q}} \subseteq \Gamma_{\mathbb{Q}}$ is a fixed point if $\iota(v_0) = v_0$. (Such v_0 always exists.)

Step 1: Construction of $\mathscr{X}$

Theorem (Caporaso, Amini–Baker–Brugallé–Rabinoff, K-Yamaki)

R: cDVR with fractional field $\mathbb{K}$ *and residue field* k, $\mathrm{ch}(k) \neq 2$

$\bar{G} = (G, w)$: *hyperelliptic vertex-weighted graph*

Assume that any vertex $v \in V(G)$ *of valence* 1 *satisfies* $w(v) > 0$.

Then the following are equivalent.

(i) *For every vertex* v *of* G, *there are at most* $(2w(v) + 2)$ *positive-type bridges emanating from* v.

(ii) *There exists a hyperelliptic nodal curve* X_0 *over* k *with dual graph* $\bar{G}$.

(iii) *There exists a regular, generically smooth, semi-stable hyperelliptic R-curve* $\mathscr{X}$ *with reduction graph* $\bar{G}$.

Remark

- The equivalence between (i) (graph with $2w(v) + 2$ condition) and (ii) (existence of hyperelliptic nodal curve X_0) is due to Caporaso.
- A version of Theorem (where $\mathbb{K}$ is not a discrete valuation field, but an algebraically closed valuation field) is shown by Amini–Baker–Brugallé–Rabinoff.
- The equivalence between (ii) (existence of hyperelliptic nodal curve X_0) and (iii) (existence of hyperelliptic R-curve) is shown by a deformation argument.

Corollary

R: cDVR with fractional field $\mathbb{K}$ and residue field k, $\mathrm{ch}(k) \neq 2$
$\bar{G} = (G, w)$: hyperelliptic vertex-weighted graph
Then the following are equivalent.

(i) *For every vertex v of G, there are at most $(2w(v) + 2)$ positive-type bridges emanating from v.*

(ii) *There exists a regular, generically smooth, semi-stable R-curve $\mathscr{X}$ with reduction graph $\bar{G}$ such that the generic fiber X is hyperelliptic.*

We are going to show that this $\mathscr{X}$ has the desired property.

Step 2: Computation of rank of reduced divisors on a hyperelliptic graph

$\bar{G} = (G, w)$: a hyperelliptic vertex-weighted graph

$\bar{\Gamma} = (\Gamma, w)$: associated metric graph of $\bar{G}$

$v_0 \in \Gamma_{\mathbb{Q}}$: a fixed point

Theorem (rank on hyperelliptic vertex-weighted graph)

For any v_0-reduced effective divisor $E \in \mathrm{Div}(\Gamma)$, we put $r = \lfloor \frac{E(v_0)}{2} \rfloor$, where $E(v_0)$ is the coefficient of E at v_0. Then

$$r_{\bar{\Gamma}}(E) = \begin{cases} r & (\textit{if } \deg(E) - r \leq g(\bar{\Gamma})) \\ \deg(E) - g(\bar{\Gamma}) & (\textit{if } \deg(E) - r \geq g(\bar{\Gamma}) + 1) \end{cases}$$

There is a corresponding formula on rank on a hyperelliptic curve.

Proposition (rank on hyperelliptic curve)

X: a hyperelliptic curve over $\mathbb{K}$ (smooth, connected)

ι_X: hyperelliptic involution on X

For any effective divisor $\widetilde{E} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$, we express $\widetilde{E}$ as

$$\widetilde{E} = P_1 + \cdots + P_r + \iota_X(P_1) + \cdots + \iota_X(P_r) + Q_1 + \cdots + Q_s,$$

where $P_1, \ldots, P_r, Q_1, \ldots, Q_s \in X(\bar{\mathbb{K}})$ and $\iota_X(Q_i) \neq Q_j$ for any $i \neq j$ with $1 \leq i, j \leq s$. Then

$$r_X(\widetilde{E}) = \begin{cases} r & (\text{if } \deg(\widetilde{E}) - r \leq g(X)), \\ \deg(\widetilde{E}) - g(X) & (\text{if } \deg(\widetilde{E}) - r \geq g(X) + 1). \end{cases}$$

Comparison of rank on a hyperelliptic vertex-weighted graph and rank on hyperelliptic curve, and Baker's specialization lemma imply the following:

$\bar{G} = (G, w)$: a hyperelliptic vertex-weighted graph

$\mathscr{X}$: regular, generically smooth, semi-stable R-curve with reduction graph $\bar{G}$ with hyperelliptic generic fiber X

If $E \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ is a v_0-reduced effective divisor, then there exists $\widetilde{E} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$ with $\tau_*(\widetilde{E}) = E$ s.t.

$$r_{\bar{\Gamma}}(E) = r_X(\widetilde{E}).$$

Step 3: Lifting of divisors

$\bar{G} = (G, w)$: a hyperelliptic vertex-weighted graph with metric graph Γ

$\mathscr{X}$: a regular, generically smooth, semi-stable R-curve with reduction graph $\bar{G}$ with hyperelliptic generic fiber X

We want to show:

$$^{\forall}D \in \mathrm{Div}(\Gamma_{\mathbb{Q}}),\ ^{\exists}\widetilde{D} \in \mathrm{Div}(X_{\bar{\mathbb{K}}}) \text{ with } D = \tau_*(\widetilde{D}) \text{ s.t. } r_{\bar{\Gamma}}(D) = r_X(\widetilde{D}).$$

Have shown: OK if D is v_0-reduced.

In general, for $D \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$, let $E \in \mathrm{Div}(\Gamma_{\mathbb{Q}})$ be the v_0-reduced divisor such that $N := D - E \in \mathrm{Prin}(\Gamma_{\mathbb{Q}})$. Let $\widetilde{E} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$ be a desired lift. Since $\tau_* : \mathrm{Prin}(X_{\bar{\mathbb{K}}}) \to \mathrm{Prin}(\Gamma_{\mathbb{Q}})$ is surjective, there exists $\widetilde{N} \in \mathrm{Prin}(X_{\bar{\mathbb{K}}})$ such that $\tau_*(\widetilde{N}) = N$. We put $\widetilde{D} := \widetilde{E} + \widetilde{N} \in \mathrm{Div}(X_{\bar{\mathbb{K}}})$. Then $\widetilde{D}$ is a desired lift. $\square$

Part 4 Relation to the algebraic rank

k: an algebraically closed field
$\bar{G} = (G, w)$: a vertex-weighted graph with associated metric graph Γ
$D \in \mathrm{Div}(G)$

As in Len's talk,

Definition (algebraic rank, Caporaso)

$$r_{\bar{G}}^{\mathrm{alg},k}(D) := \max_{X_0} \left\{ \min_{E} \left\{ \max_{\mathscr{E}_0} r_{X_0}(\mathscr{E}_0) \right\} \right\},$$

where X_0 runs through all connected reduced nodal curves over k with dual graph $\bar{G}$, $E \in \mathrm{Div}(G)$ runs through all divisors with $E \sim D$, and $\mathscr{E}_0 \in \mathrm{Div}(X_0)$ runs through all Cartier divisors on X_0 with $\deg(\mathscr{E}_0|_{C_v}) = E(v)$ for any $v \in V(G)$.

Result

Theorem (algebraic rank for hyperelliptic or genus 3 graphs)

Assume one of the following.

(i) $\bar{G}$ *is hyperelliptic and* $\mathrm{ch}(k) \neq 2$*;*

(ii) $\bar{G}$ *is non-hyperelliptic and of genus* 3*.*

Then, for any $D \in \mathrm{Div}(G)$*, we have* $r^{\mathrm{alg},k}_{\bar{G}}(D) \geq r_{\bar{G}}(D)$*.*

Remark

- Here $r_{\bar{G}}(D) := r_{\bar{\Gamma}}(D)$, where Γ is the associated metric graph of G.
- Caporaso–Len–Melo prove that the other direction $r^{\mathrm{alg},k}_{\bar{G}}(D) \leq r_{\bar{G}}(D)$ holds for any vertex-weighted graph $\bar{G}$ and $D \in \mathrm{Div}(G)$. Thus for a hyperelliptic or a genus 3 graph, the equality $r^{\mathrm{alg},k}_{\bar{G}}(D) = r_{\bar{G}}(D)$ holds.

Remark (continued)

- Caporaso–Len–Melo give many other examples of graphs with the equality $r_{\bar{G}}^{\mathrm{alg},k}(D) = r_{\bar{G}}(D)$.
- They also show that there exist a graph $\bar{G}$ and a divisor $D \in \mathrm{Div}(G)$ such that $r_{\bar{G}}^{\mathrm{alg},k}(D) < r_{\bar{G}}(D)$.

Outline of the proof of Theorem

1. A variant of the question on lifting of divisors
2. Relation between lifting divisors and algebraic rank
3. Decomposition of graphs with bridges

Step 1: A variant of the question

R: cDVR with $\mathrm{Frac}(R) = \mathbb{K}$ and residue field k

$\bar{G} = (G, w)$: a vertex-weighted graph

Question (a variant)

Does there exist a regular, generically smooth, semi-stable R-curve $\mathscr{X}$ with reduction graph $\bar{G}$ satisfying the following condition (F)?

> ${}^{\forall} D \in \mathrm{Div}(G)$, ${}^{\exists} \widetilde{D} \in \mathrm{Div}(X)$ with $D = \rho_*(\widetilde{D})$ s.t.
>
> $$r_{\bar{G}}(D) = r_X(\widetilde{D}).$$

Here, $\rho_* : \mathrm{Div}(X) \to \mathrm{Div}(G)$ is the specialization map.

(Comparison to the original question: D in $\mathrm{Div}(G)$ not in $\mathrm{Div}(\Gamma_{\mathbb{Q}})$; $\widetilde{D}$ in $\mathrm{Div}(X)$ not in $\mathrm{Div}(X_{\bar{\mathbb{K}}})$; and the specialization map is ρ_* not $\tau_* : \mathrm{Div}(X_{\bar{\mathbb{K}}}) \to \mathrm{Div}(\Gamma_{\mathbb{Q}})$.)

Step 2: Relation between Question (a variant) and algebraic rank

Caporaso kindly told us relation between Question (a variant) and algebraic rank as follows.

Proposition

R: cDVR with $\mathrm{Frac}(R) = \mathbb{K}$ and residue field k

$\bar{G} = (G, w)$: a vertex-weighted graph

$\mathscr{X}$: a regular, generically smooth, semi-stable R-curve with reduction graph $\bar{G}$ with generic fiber X

Assume that $\mathscr{X}$ satisfies the condition (F).

Then, for any divisor $D \in \mathrm{Div}(G)$, we have

$$r_{\bar{G}}^{\mathrm{alg},k}(D) \geq r_{\bar{G}}(D).$$

Proof (Caporaso)

Recall

$$r^{\mathrm{alg},k}_{\bar{G}}(D) := \max_{X_0}\left\{\min_{E}\left\{\max_{\mathscr{E}_0} r_{X_0}(\mathscr{E}_0)\right\}\right\}.$$

If such $\mathscr{X}$ exists, we let $X_0 = \mathscr{X}_k$.
For any $E \in \mathrm{Div}(G)$ with $D \sim E$, we take a lift $\widetilde{E} \in \mathrm{Div}(X)$ preserving the rank. Let $\mathscr{E}$ be the Zariski closure of $\widetilde{E}$ in $\mathscr{X}$, and we put $\mathscr{E}_0 := \mathscr{E}|_{X_0}$. Then

$$r^{\mathrm{alg},k}_{\bar{G}}(D) \geq r_{X_0}(\mathscr{E}_0) \overset{(*)}{\geq} r_X(\widetilde{E}) \overset{(**)}{=} r_{\bar{G}}(E) = r_{\bar{G}}(D),$$

where (*) is deduced from the upper-semicontinuity of the cohomology, and (**) is the condition (F). □

R: cDVR with $\mathrm{Frac}(R) = \mathbb{K}$ and residue field k

$\bar{G} = (G, w)$: a vertex-weighted graph

As in the original question, we have the following theorem.

Theorem

Assume one of the following.

(i) $\bar{G}$ *is hyperelliptic and* $\mathrm{ch}(k) \neq 2$*; Further, for each* $v \in V(G)$*, there are at most* $(2w(v) + 2)$ *positive-type bridges emanating from* v*.*

(ii) $\bar{G}$ *is non-hyperelliptic and of genus* 3*.*

Then there exists a regular, generically smooth, semi-stable R*-curve* $\mathscr{X}$ *with reduction graph* $\bar{G}$ *satisfying the condition (F) (i.e. the answer to Question (a variant) is YES.)*

Thus we have, for any $D \in \mathrm{Div}(G)$,

$$r^{\mathrm{alg},k}_{\bar{G}}(D) \geq r_{\bar{G}}(D)$$

provided that $\bar{G}$ is one of the following.

(i) $\bar{G}$ is hyperelliptic and $\mathrm{ch}(k) \neq 2$; Further, for each $v \in V(G)$, there are at most $(2w(v) + 2)$ positive-type bridges emanating from v.

(ii) $\bar{G}$ is non-hyperelliptic and of genus 3.

We want to delete the $(2w(v) + 2)$ condition for the hyperelliptic case.

Step 3: Decomposition of graphs with bridges

$\bar{G} = (G, w)$: a vertex-weighted graph having a bridge e with endpoints v_1, v_2

G_i $(i = 1, 2)$: connected components of $G \setminus \{e\}$ with $v_i \in V(G_i)$

$\bar{G}_i := (G_i, w|_{V(G_i)})$

Proposition

For any $D \in \mathrm{Div}(G)$, let $D_i \in \mathrm{Div}(G_i)$ be the restriction of D to G_i. Then

$$r_{\bar{G}}(D) \leq \begin{cases} r_{\bar{G}_1}(D_1) + r_{\bar{G}_2}(D_2) + 1 & (\text{if } v_i \in \mathrm{Bs}(|D_i|) \text{ for } {}^{\forall} i = 1, 2), \\ r_{\bar{G}_1}(D_1) + r_{\bar{G}_2}(D_2) & (\text{otherwise}). \end{cases}$$

Remark

If $\bar{G}$ is hyperelliptic, then $\bar{G}_i$ is hyperelliptic (or of genus 0 or 1).

There is a corresponding formula on nodal curves.

Lemma

Let X be a nodal curve. We assume that X has a decomposition as $X = X_1 \cup X_2$ into two nodal curves so that X_1 and X_2 meet at exactly one point p. Let $\widetilde{D}$ be a Cartier divisor on X, and we set $\widetilde{D}_i = \widetilde{D}\Big|_{X_i} \in \mathrm{Div}(X_i)$ for $i = 1, 2$. Then

$$r_X(\widetilde{D}) = \begin{cases} r_{X_1}(\widetilde{D}_1) + r_{X_2}(\widetilde{D}_2) + 1 & (\textit{if } p \in \mathrm{Bs}(|\widetilde{D}_i|) \textit{ for } {}^\forall i = 1, 2), \\ r_{X_1}(\widetilde{D}_1) + r_{X_2}(\widetilde{D}_2) & (\textit{otherwise}). \end{cases}$$

Combining Proposition and Lemma, we obtain $r_{\bar{G}}^{\mathrm{alg},k}(D) \geq r_{\bar{G}}(D)$ for any hyperelliptic graph $\bar{G}$ (without the $(2w(v)+2)$ condition). □