Toric Geometry and Dyson's Lemma

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(work in progress)

Abstract. In 1989, I proved a Dyson lemma for products of two smooth projective curves of arbitrary genus. In 1995, M. Nakamaye extended this to a result for a product of an arbitrary number of smooth projective curves of arbitrary genus, in a formulation involving an additional "perturbation divisor." In 1998, he also found an example in which a hoped-for Dyson lemma is false without such a perturbation divisor. This talk will present work in progress on eliminating the perturbation divisor by using a different definition of "volume" at the points under consideration. The proof involves toric and toroidal geometry, and this is reflected in the statement as well.

§1. Background (Historical)

Theorem (Roth). *Let* $\alpha \in \overline{\mathbb{Q}}$, *let* $\epsilon > 0$, *and let* $C \in \mathbb{R}$. *Then there are only finitely many* $p/q \in \mathbb{Q}$ $(p, q \in \mathbb{Z}, \ \gcd(p,q) = 1)$ *such that*

$$\left| \frac{p}{q} - \alpha \right| \le \frac{C}{|q|^{2+\epsilon}} .$$

1909	Thue	$\frac{d}{2} + 1 + \epsilon \qquad (d = [\mathbb{Q}(\alpha) : \mathbb{Q}])$
1921	Siegel	$\min\left\{ \frac{d}{s+1} + s : 0 \le s < d \right\} + \epsilon$
1947	Dyson	$\sqrt{2d} + \epsilon$
1952	Gel'fond	$\sqrt{2d} + \epsilon$?
1955	Roth	$2 + \epsilon$

Sketch of method:

(1). Assume that there are infinitely many p/q satisfying the inequality.

(2). Choose 2 ($<$ Roth) or n (Roth) of these p/q, satisfying certain conditions: $p_1/q_1, \dots, p_n/q_n$.

(3). Construct an auxiliary polynomial P in n variables based on $q_1, \dots, q_n$. The coefficients of P should lie in $\mathbb{Z}$, and their absolute values should be bounded.

(4). Use the bounds on the coefficients of P, together with the approximation condition on each p_i/q_i, to give a positive lower bound for the vanishing of P at $(p_1/q_1, \dots, p_n/q_n)$.

(5). Derive a contradiction, using either "Roth's lemma" or "Dyson's lemma."

Gel'fond used a Roth-like lemma, as opposed to a Dyson-like one.

§2. Detailed Description of Dyson's Lemma

Let $C_1, \dots, C_n$ be smooth projective curves over $\mathbb{C}$, let Y be an effective divisor on $C_1 \times \cdots \times C_n$, and let $d_i = (Y \,.\, \widetilde{C}_i)$ for all i, where $\widetilde{C}_i$ is a fiber of the map $C_1 \times \cdots \times C_n \to \prod_{j \ne i} C_j$. Assume that $d_i > 0$ for all i.

Definition. For $P \in C_1 \times \cdots \times C_n$ define the index of Y at P relative to $\mathbf{d} = (d_1, \dots, d_n)$ as

$$t_{\mathbf{d},Y}(P) = \min \left\{ \frac{i_1}{d_1} + \cdots + \frac{i_n}{d_n} : \left(\frac{\partial}{\partial z_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{i_n} f(P) \ne 0 \right\} ,$$

where f is a local defining equation for Y at P and z_i is a local coordinate on C_i for each i.

We also define $\operatorname{Vol}(t)$ as

$$\operatorname{Vol}(t) = \text{volume of } \left\{ (x_1, \dots, x_n) \in [0,1]^n : \sum x_i \le t \right\} .$$

Question. *Given $C_1, \dots, C_n, Y, d_1, \dots, d_n$ as above, and points $P_1, \dots, P_s \in \prod C_i$ lying in distinct fibers over C_i for all i, can one show that*

$$\sum_{i=1}^{s} \operatorname{Vol}(t_{\mathbf{d},Y}(P_i)) \le \frac{1}{d_1 \cdots d_n} \cdot \frac{(Y^n)}{n!} + O\left(\max \left\{ \frac{d_j}{d_i} : i < j \right\} \right)$$

with the constant in $O(\cdot)$ depending only on $g(C_1), \dots, g(C_n)$, n, s?

The intuition behind this is that generally

$$h^0 \left(\prod C_i, Y \right) \approx \frac{(Y^n)}{n!}$$

(if Y is ample), and $d_1 \cdots d_n \cdot \operatorname{Vol}(t_{\mathbf{d},Y}(P_i))$ is the approximate number of linear conditions one would use to naïvely achieve the given index at P_i. Thus, the inequality becomes best possible in the limit as $\max\{d_j/d_i\} \to 0$.

More History

n	C_i	s	
2	$\mathbb{P}^1$	any	Dyson 1947 (some differences)
2	$\mathbb{P}^1$	any	Viola 1985
any	$\mathbb{P}^1$	any	Esnault-Viehweg 1984; Roth proof
2	any	any	V. 1989; new proof of Mordell
any	any	0	V. 1990 (unpublished)
any	any	any	Nakamaye 1995 "perturbation divisor"
counterexample			Nakamaye 1998

§3. Proofs when $n \le 2$

When $n = 1$ (simple but instructive):

$$\sum_i \frac{\deg_{P_i}(Y)}{d_1} \le \frac{\deg Y}{d_1} .$$

No $O(\cdot)$ term

When $n = 2$ and $s = 0$: Let Y be an effective divisor on $C_1 \times C_2$.

If Y is a prime divisor not contained in a fiber over C_1, then positivity of the relative dualizing sheaf ω_{Y/C_1} gives

$$Y \,.\, (Y + \mathrm{pr}_2^* K_{C_2}) = \deg \omega_{Y/C} \ge 0 ;$$

therefore

$$\begin{aligned} Y^2 &\ge -(2g(C_2) - 2)(Y \,.\, \widetilde{C}_2) \\ &\ge -\max\{2g(C_2) - 2, 0\} d_2 , \end{aligned}$$

where $\widetilde{C}_2 = \{\text{pt.}\} \times C_2 \subseteq C_1 \times C_2$, and $d_2 = Y \,.\, \widetilde{C}_2$. The above inequality is also true if Y is a fiber over C_1.

In general, $Y = \sum e_i Y_i$ with Y_i distinct prime divisors, and

$$\begin{aligned} Y^2 &\ge \sum e_i^2 Y_i^2 \\ &\ge -\max\{2g(C_2) - 2, 0\} \sum e_i^2 (Y_i \,.\, \widetilde{C}_2) \\ &\ge -\max\{2g(C_2) - 2, 0\} \max\{e_i\} \sum (e_i Y_i \,.\, \widetilde{C}_2) \\ &\ge -\max\{2g(C_2) - 2, 0\} \cdot d_2 \cdot d_2 . \end{aligned}$$

Thus

$$\frac{Y^2}{2d_1 d_2} \ge -\frac{d_2}{2d_1} \max\{2g(C_2) - 2, 0\} .$$

(In applications, d_2/d_1 is small.)

When $n = 2$ and $s = 1$:

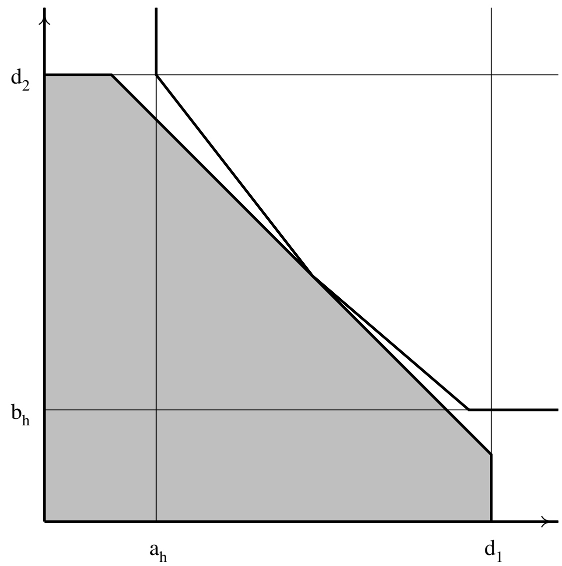
Pick a point $P_i \in C_i$ for each $i = 1, 2$. Let $\pi\colon X \to C_1 \times C_2$ be the blowing-up at (P_1, P_2), let E be the exceptional divisor, and let e be the multiplicity of Y at (P_1, P_2). Then $\pi^* Y - eE =: Y'$ is the strict transform, and the above method gives

$$\frac{Y'^2}{2d_1 d_2} \ge -\frac{d_2}{2d_1} \max\{2g(C_2) - 2 + 1, 0\} ;$$

therefore

$$\frac{Y^2}{2d_1d_2} \geq \frac{e^2}{2d_1d_2} - \frac{d_2}{2d_1}\max\{2g(C_2)-2+1,0\}\ .$$

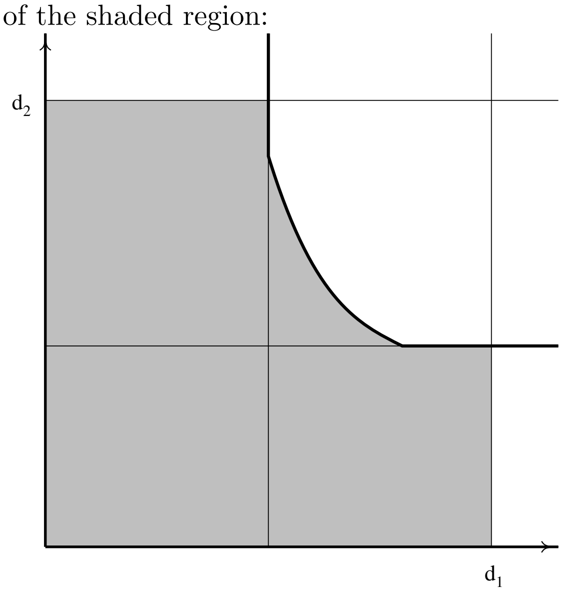
Of course, $e^2/2d_1d_2$ is not Vol of the index:



However, we can do additional blowings-up to get

$$\frac{Y^2}{2d_1d_2} \geq A - \frac{d_2}{2d_1}\max\{2g(C_2)-2+1,0\}\ ,$$

where A is the area of the shaded region:

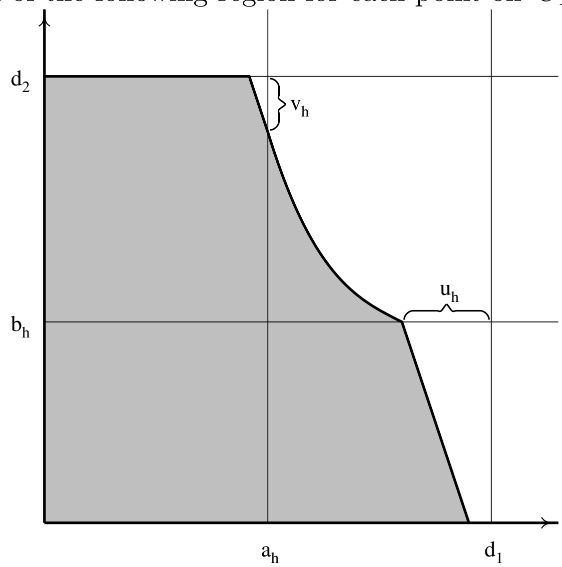


When $n = 2$ and s is arbitrary:

Additional combinatorial arguments give:

$$\frac{Y^2}{2d_1d_2} \geq \sum A_i - \frac{d_2}{2d_1} \max\{2g(C_2) - 2 + s, 0\} ,$$

where A_i is the area of the following region for each point on $C_1 \times C_2$:



Why is the (upper) region cut off?

(a). It's false otherwise (counterexample: $x_1^{d_1}x_2^{d_2}$ on $\mathbb{P}^1 \times \mathbb{P}^1$).

(b). You get the rest "for free," so they shouldn't count.

§4. A First Proposal for Vol when $n = 3$

For $t \in \mathbb{R}$ let $\operatorname{Vol}_{\mathscr{O}(Y),\mathbf{d}}(t)$ be the volume of the set

$$\begin{aligned}\{(x,y,z)\in[0,\infty)^3 : {} & x\le 1,\ y\le 1,\ z\le 1,\\ & x+y\le t_{12},\ x+z\le t_{13},\ y+z\le t_{23},\\ & x+y+z\le t\}\end{aligned}$$

where t_{12} satisfies

$$\operatorname{Vol}_{\mathscr{O}(Y)\big|_{F_3},(d_1,d_2)}(t_{12}) = \frac{(Y^2\,.\,F_3)}{2d_1d_2}$$

and t_{13} , t_{23} are defined similarly; d_1, d_2, d_3 are as defined earlier; and F_i is a fiber of $C_1 \times C_2 \times C_3 \to C_i$.

For $n > 3$: you can see a pattern.

Why hasn't this come up before???

(a). It has ($n = 1$).
(b). When $n = 2$: no change
(c). When $n > 2$ and $C_i = \mathbb{P}^1$ for all i (say $n = 3$),
$\mathscr{O}(Y) \cong \mathscr{O}(d_1, d_2, d_3)$, $(Y^2\,.\,F_3) = 2d_1d_2$, so $\operatorname{Vol}(t_{12}) = 1$, giving $t_{12} = 2$, etc.

Also, this definition addresses Nakamaye's counterexample.

It also fits in with the principle of not giving credit for things that are free, *including when you apply Dyson's lemma to linear subspaces parallel to coordinate subspaces.*

The conjecture (still unproved, but may be close) is that, given points $P_i \in C_i$ for $i = 1, 2, 3$, if Y is an effective divisor on $C_1 \times C_2 \times C_3$, and if t is the index of Y at (P_1, P_2, P_3) with respect to (d_1, d_2, d_3) , then

$$\operatorname{Vol}_{\mathscr{O}(Y),\mathbf{d}}(t) \le \frac{Y^3}{6d_1d_2d_2} - O\left(\max\left\{\frac{d_j}{d_i} : i < j\right\}\right) .$$

It should also be possible to extend this result to more than one point.

§5. Toric and Toroidal Geometry

If $C_i \cong \mathbb{P}^1$ for all i, then the blowing-up mentioned for the proof when $n = 2$ and $s = 1$ is a toric variety.

A toric variety of dimension n can be described by a fan in $\mathbb{R}^n$:

Definition. A fan Σ in $N_{\mathbb{R}} := \mathbb{R}^n$ is a set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$, such that:

(i). every face of a cone in Σ is a cone in Σ, and
(ii). the intersection of every two cones in Σ is a cone in Σ.

What if $g(C_i) > 0$?

Definition. A toroidal embedding is an open immersion $U \hookrightarrow X$ that locally analytically looks like the open immersion of $\mathbb{G}_{\mathrm{m}}^n$ into a toric variety of dimension n (as its dense orbit).

Definition ([**K-K-M-S**], Ch. II, § 1, Def. 1). Let k be a field.

(a). A **toroidal embedding** is an open immersion $U \hookrightarrow X$, where X is a normal variety of dimension n over k and U is a smooth open subset of X, such that the following condition holds. For every closed point $x \in X$ there is an n-dimensional affine toric variety X_σ with dense orbit T, a point $t \in X_\sigma$, and a local isomorphism

$$\phi\colon \widehat{\mathscr{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathscr{O}}_{X_\sigma,t}$$

of local k-algebras that takes the ideal corresponding to $X \setminus U$ in $\widehat{\mathscr{O}}_{X,x}$ to the ideal corresponding to $X_\sigma \setminus T$ in $\widehat{\mathscr{O}}_{X_\sigma,t}$.

(b). Let $U \hookrightarrow X$ be a toroidal embedding. A **local model** at a closed point $x \in X$ is a triple (X_σ, t, ϕ) as in part (a), such that the orbit of t is closed in X_σ.

(c). A toroidal embedding $U \hookrightarrow X$ is **without self-intersection** if the irreducible components of $X \setminus U$ are normal.

For the application to Dyson's lemma, the following example is key.

Example.

(a). Let X be a smooth curve over $k = \mathbb{C}$, and let U be a nonempty open subset. Then $U \hookrightarrow X$ is a toroidal embedding without self-intersection.

(b). A product of finitely many toroidal embeddings without self-intersection is again a toroidal embedding without self-intersection.

In this talk, all toroidal embeddings will be assumed to be without self-intersection.

In place of a fan, a toroidal embedding is described by a conical polyhedral complex with integral structure, whose ambient space is a topological space on which the multiplicative group $\mathbb{R}_{>0}$ acts.

I'll just call it a fanoid.

Example. Let X be a curve.

(a). For the toroidal embedding $X\setminus\{\text{five points}\}\hookrightarrow X$, the above ambient space looks like: [draw].

(b). For the toroidal embedding $X\setminus\{\text{two points}\}\hookrightarrow X$, the above ambient space can be identified with $\mathbb{R}$.

For Dyson's lemma, our setup will be the following.

For each i let P_i and Q_i be distinct closed points in C_i, and let $U_i = C_i\setminus\{P_i, Q_i\}$. Let $U=\prod U_i$. Then $U\hookrightarrow\prod C_i$ is a toroidal embedding, whose fanoid is the orthant fan in $\mathbb{R}^n$.

Let $P=(P_1,\dots,P_n)\in\prod C_i$.

Given an effective Cartier divisor Y on $\prod C_i$, we can associate to Y an element of $\widehat{\mathscr{O}}_{\prod C_i,P}$, well-defined up to a multiplicative unit, and a well-defined Newton polyhedron in $[0,\infty)^n$.

Proposition. *Let Y be an effective divisor on $\prod C_i$, and assume that $Y\,.\,C_i>0$ for all i. Let Δ_0 be the fanoid associated to $U\hookrightarrow\prod C_i$. Then there is a refinement Δ of Δ_0 and an effective divisor Y' on the allowable modification X of $\prod C_i$ associated to Δ, such that the following statements are true.*

(i). *The support of Y' does not contain any stratum of X for which the closure of the image in $\prod C_i$ contains P.*

(ii). *The divisor $E:=\pi^*Y-Y'$ is effective, and its support is a union of closures Z of strata that satisfy $\pi(Z)\ni P$.*

(iii). *Let f be a locally defining function for Y on $\prod C_i$ near P, and let Π_f be the associated Newton polyhedron (as above). Recall that for admissible $x\in\mathbb{R}^n$, the expression*

$$\operatorname{Vol}\left(\left(\prod_{i=1}^n[0,x_i]\right)\setminus\Pi_{f,x}\right)$$

is a polynomial in $x_1,\dots,x_n$ of degree 1 in each x_i. For each $I\subseteq\{1,\dots,n\}$ let c_I denote the coefficient of $\prod_{i\in I}x_i$ in this polynomial. Then

$$\frac{Y^n-(Y')^n}{n!}=\sum_{I\subsetneq\{1,\dots,n\}}c_I\frac{Y^{\#I}\,.\,C_I}{(\#I)!}\,.$$

Theorem (Dyson's lemma for one point (so far)). *Let Y be an effective divisor on $\prod_{i=1}^n C_i$, and let P be a closed point on $\prod C_i$. For each i let $d_i=Y\,.\,C_i$ and $P_i=\operatorname{pr}_i(P)$ (here pr_i denotes the projection to the i^{th} factor). Let K_i be* *a canonical divisor of* C_i , *and let* K_i^+ *be the divisor* $-P_i$ *if* $C_i \cong \mathbb{P}^1$, *or* K_i *otherwise. Then*

$$\begin{aligned}
&\frac{1}{d_1 \cdots d_n} \sum_{I \subsetneqq \{1,\dots,n\}} c_I \frac{Y^{\#I} \,.\, C_I}{(\#I)!} \\
&\qquad \le \frac{Y \,.\, (Y + \sum_{i=1}^n (d_{i+1} + \cdots + d_n) \operatorname{pr}_i^*(K_i^+ + P_i))^{n-1}}{n! \cdot d_1 \cdots d_n} \\
&\qquad\qquad + O\left(\max\left\{ \frac{d_j}{d_i} : i < j \right\} \right) \ .
\end{aligned}$$

The proof basically involves showing that the divisor

$$Y' + \sum_{i=1}^n (d_{i+1} + \cdots + d_n) \operatorname{pr}_i^*(K_i^+ + P_i)$$

is nef.

§6. More Details on the Proof

[This section was omitted from the talk due to time constraints.]

First of all, it is possible to ensure conditions (i) ($\operatorname{Supp} Y'$ does not contain any stratum ...) and (ii) (the divisor E) by standard results in toric or toroidal geometry.

For condition (iii) (on $Y^n - (Y')^n$): In the toric case it follows from standard results in toric geometry, since the right-hand side of the equation is just the polynomial expressing the Euclidean volume. In the toroidal case it is shown by comparing with the results in the toric case.

To show the nefness part, let's consider the $s = 0$ case. In this case we want to show that the divisor

$$Y + \sum_{i=1}^{n} (d_{i+1} + \cdots + d_n) \operatorname{pr}_i^* K_i^+$$

on $\prod C_i$ is nef. Here K_i^+ is taken to be trivial if $g(C_i) = 0$. Let Z be a curve in $X := \prod C_i$. In the special case in which Z is nonsingular, we have an injection

$$\mathscr{C}_{Z/X} \hookrightarrow \bigoplus_{j=1}^{n} \operatorname{pr}_i^* \Omega_{C_i/k} .$$

We use this in place of positivity of the dualizing sheaf in the $n = 2$ case. If a component of Y contains Z, then we blow up along Z and replace X with the exceptional divisor of the blow-up. This reduces $\dim X$ by 1, and we can proceed by induction.

If Z is singular, then we blow up X at closed points until the strict transform of Z becomes nonsingular, and then a similar argument works.

Finally, if $s = 1$, then start with X equal to the toroidal modification mentioned above (plus blow up points to desingularize Z) to again get nefness.

The idea of using nefness in Dyson's lemma was first used by Esnault and Viehweg.

When $n > 2$ the hardest part is bounding the multiplicities of Y along Z.