Integral points on varieties and Schmidt's Subspace Theorem

Aaron Levin

Michigan State University

Vojta's Conjectures
BIRS, Banff

Diophantine Equations

- Classically, we are interested in the study of solutions to a system of polynomial equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0, \\ f_2(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ f_m(x_1, \ldots, x_n) &= 0, \end{aligned}$$

  where $x_1, \ldots, x_n$ lie in a ring R of arithmetic interest.
- Here, we'll be interested in:
  - $R = \mathbb{Q}$, or more generally a number field k $\left(\text{e.g., } \mathbb{Q}\left(\sqrt{2}\right)\right)$
  - $R = \mathbb{Z}$, or more generally a ring of integers $\mathcal{O}_k$ or S-integers $\mathcal{O}_{k,S}$ of a number field k $\left(\text{e.g, } \mathbb{Z}\left[\sqrt{2}\right], \mathbb{Z}\left[\frac{1}{3}\right]\right)$.

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Integral points on varieties

- The system of polynomial equations defines a geometric object in affine space or projective space.
- The study of integral or rational solutions becomes the study of integral and rational points on varieties.
- Suppose that X is an affine variety over a number field k.
- For any affine embedding $X \subset \mathbb{A}^N$ we can define the set

$$X(\mathcal{O}_{k,S}) = \{(x_1, \ldots, x_N) \in X \mid x_1, \ldots, x_N \in \mathcal{O}_{k,S}\}$$

  of S-integral points.
- This depends on the affine embedding. However:
- S-integral points in one affine embedding will be S'-integral in another affine embedding for some larger finite S'.

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Integral and rational points

- Alternate definitions of integral points:
  - Scheme-theoretic definition (depends on choice of integral model over $\mathcal{O}_{k,S}$).
  - Definition using (local) height functions.
- These can be used to define integral points for arbitrary varieties.
- When we write $X(\mathcal{O}_{k,S})$ for X affine we will always assume some fixed affine embedding $X \subset \mathbb{A}^N$.
- We will frequently write X as $X = \tilde{X} \setminus D$, where D is an effective divisor on $\tilde{X}$.
- We also have the set of rational points $X(k)$.
- When $X \subset \mathbb{A}^N$ (or $\mathbb{P}^N$) this is the set of points of X with coordinates in k (or with some choice of projective coordinates all in k).

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Siegel's and Falting's Theorems for Curves

Rational Points on Curves

- For a curve there is a fundamental geometric invariant: the genus.
- The fundamental theorem on rational points on curves is Faltings' theorem (Mordell's conjecture):

Theorem (Faltings)

Let C be a curve defined over a number field k of genus $g(C) \geq 2$. Then the set of k-rational points $C(k)$ is finite.

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- The fundamental result for integral points on curves is a classical theorem of Siegel.

Theorem (Siegel)

Let $C \subset \mathbb{A}^n$ be an affine curve defined over k. Let $\tilde{C}$ be a projective closure of C. If

- *$\tilde{C}$ has positive genus*

or

- *C is rational with more than two points at infinity ($\#\tilde{C} \setminus C \geq 3$)*

then the set of integral points $C(\mathcal{O}_k)$ is finite.

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An example

- Consider the rational affine curve C defined by $x^2 - 3y^2 = 1$.
- We have $C \subset \widetilde{C}$, where $\widetilde{C}$ is the projective plane curve $\widetilde{C} : x^2 - 3y^2 = z^2$.
- The points at infinity $\widetilde{C} \setminus C$ correspond to the points on $\widetilde{C}$ with $z = 0$. There are two such points $[x : y : z] = [\pm\sqrt{3} : 1 : 0]$.
- So Siegel's theorem does not apply.
- C does in fact have infinitely many $\mathbb{Z}$-integral points. C is defined by a so-called Pell equation. If $n \in \mathbb{N}$,

$$x + \sqrt{3}y = (2 + \sqrt{3})^n,$$

then (x, y) will be an integral point on C.

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Reformulation of Siegel's theorem

- Recall that Siegel's theorem states that an affine curve C over a number field k has only finitely many integral points if either:
  - C has positive genus.
  - C is rational and has three or more points at infinity.
- In particular, we have the (apparently) weaker statement:

Theorem (Siegel)

Let $C \subset \mathbb{A}^n$ be an affine curve over k. Let $\tilde{C}$ be a projective closure of C. If C has more than two points at infinity ($\#\tilde{C} \setminus C > 2$) then the set of integral points $C(\mathcal{O}_k)$ is finite.

- In fact, using unramified coverings, one can derive Siegel's theorem in full generality from this weaker statement.

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Another Viewpoint

- Siegel's theorem: If C is an affine curve with sufficiently many points at infinity then C has only finitely many integral points.
- How does this concept generalize to higher dimensions?
- General setup:
  - $\tilde{X}$ is a nonsingular projective variety over a number field k.
  - S a finite set of places of k.
  - $D = \sum_{i=1}^{r} D_i$ is a divisor on $\tilde{X}$ with the D_i effective divisors.
  - $X = \tilde{X} \setminus D$.
- Can we say something about integral points on X if r is large (i.e., X has many components at infinity)?
- Two things special to curves:
  - Every nontrivial effective divisor is ample
  - Distinct subvarieties (points) don't intersect.

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- Siegel's theorem: If C is an affine curve with sufficiently many points at infinity then C has only finitely many integral points.
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Higher-dimensional conjectures

Pathologies in higher dimensions

- Let $X = \tilde{X} \setminus D$ be any surface with $X(\mathcal{O}_{k,S})$ Zariski dense.
- By blowing up rational points of D, we obtain a divisor D' on $\tilde{X}'$ with arbitrarily many components and

$$X = \tilde{X} \setminus D = \tilde{X}' \setminus D'.$$

- So $\tilde{X}' \setminus D'$ admits a Zariski dense set of integral points.
- D' can have arbitrarily many components.
- So, with absolutely no assumptions, can't get a theorem like for curves.
- The exceptional curves in D' are as far as possible from being ample.
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- As a corollary of deep results on integral points on semiabelian varieties, Vojta proved:

Theorem (Vojta)

Let $\rho(\tilde{X})$ denote the Picard number of $\tilde{X}$. Suppose that D is a sum of $r > \dim \tilde{X} + \rho(\tilde{X}) - h^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ distinct irreducible components. Then $X(\mathcal{O}_{k,S})$ is not Zariski dense.

- A weaker (but related) theorem can also be proved using S-unit equations (Vojta).
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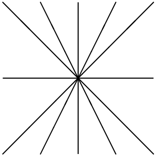
Let $X = \mathbb{P}^2$ and $D =$ 

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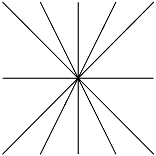
- Conclusion: Even with ample components, to obtain a finiteness result, one must take into account the number of components of D meeting at a point.

Let $X = \mathbb{P}^2$ and $D =$ 

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Higher-dimensional Siegel Conjecture

- Assuming now ampleness and general position:

Conjecture

Let $X = \widetilde{X} \setminus D \subset \mathbb{A}^N$ be an affine variety of dimension n with r ample effective divisors in general position at infinity. Then

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- Let $D = \sum_{i=1}^{r} D_i$ be a divisor on $\tilde{X}$ with D_i ample for all i.
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- Then $X = \tilde{X} \setminus D$ is of logarithmic general type.
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- For part (2), we note that a curve and an ample effective divisor always intersect.
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Known Results

Work of Corvaja and Zannier

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- This technique was subsequently developed by them for surfaces.
- They proved that if $D = \sum_{i=1}^{r} D_i$ is an ample divisor on $\widetilde{X}$ and the D_i satisfy certain intersection number inequalities, then the set of integral points $(\widetilde{X} \setminus D)(\mathcal{O}_{k,S})$ is not Zariski dense in $\tilde{X}$.

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- Building on this work, we have the following generalization of Siegel's theorem to surfaces:

Theorem (L.)

Let $X \subset \mathbb{A}^N$ be an affine surface with r ample effective divisors in general position at infinity.

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Threefolds and Beyond

- In dimensions $n > 2$, I showed that the integral points on $X = \widetilde{X} \setminus D$ are not Zariski dense if D is a sum of $r > 2n[\frac{n+1}{2}]$ ample divisors in general position.
- This was improved by Autissier ($r \geq n^2$), Corvaja, L., Zannier ($r > n^2 - n$), and by Autissier:

Theorem (Autissier)

Let $X \subset \mathbb{A}^N$ be an affine variety of dimension $n \geq 3$ with $r \geq 2n$ ample effective divisors in general position at infinity. Then $X(\mathcal{O}_{k,S})$ is not Zariski dense in X.

- For Zariski density this is the current best general result (recall the conjecture is $r \geq n + 2$).
- All the above results actually show the much stronger statement: there is a geometric set Z, independent of k and S, such that $X(\mathcal{O}_{k,S}) \setminus Z$ is finite.

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The Subspace Theorem

- The arithmetic tools used in the proofs come from Diophantine approximation.
- The most famous result here is Roth's theorem:

Theorem (Roth)

Given any algebraic number α and any $\epsilon > 0$, there are only finitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ satisfying

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Reformulation in terms of local heights

Definition

For k a number field, $\alpha \in k$, and v a place of k, define the local height (or local Weil function) with respect to α by

$$\lambda_{\alpha,v}(x) = \log \max \left\{ \frac{1}{|x-\alpha|_v}, 1 \right\}, \quad \forall x \in k, x \neq \alpha.$$

- This measures how v-adically close x is to α
- This is a local height in the sense that

$$\sum_v \lambda_{\alpha,v}(x) = h(x) + O(1).$$

- Roth's theorem: for distinct $\alpha_1, \ldots, \alpha_q \in k$, the inequality

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- Viewing $\bar{\mathbb{Q}} \subset \mathbb{A}^1 \subset \mathbb{P}^1$, we can view Roth's theorem as a statement about approximating points in $\mathbb{P}^1$.
- From this point of view, in 1970 Schmidt gave a deep generalization of Roth's theorem to the setting of approximation of hyperplanes in projective space.

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Let k be a number field and v a place (absolute value) of k. For a hyperplane $H \subset \mathbb{P}^n$ defined by the linear form $L(x_0, \ldots, x_n)$ define a local Weil function for H, with respect to v, by

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Theorem (Schmidt's Subspace Theorem)

Let k be a number field. Let S be a finite set of places of k. For each $v \in S$, let $H_{0v}, \ldots, H_{nv}$ be hyperplanes over k in $\mathbb{P}^n$ in general position. Let $\epsilon > 0$. Then there exists a finite union of hyperplanes $Z \subset \mathbb{P}^n$ such that

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- Let $\tilde{X}$ be a projective variety, D an effective divisor on $\tilde{X}$, and $X = \tilde{X} \setminus D$. Let $R = X(\mathcal{O}_{k,S})$.
- We find a morphism $\phi : \tilde{X} \to \mathbb{P}^N$, with $\phi(\tilde{X})$ not contained in any hyperplane, and a set of hyperplanes $H_{0v}, \ldots, H_{Nv}$, for each $v \in S$, such that the hyperplanes closely approximate points of $\phi(R)$.
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So things naturally breaks into two problems:

- Finding a good map $\phi : \tilde{X} \to \mathbb{P}^N$.
  - Look at the map associated to $|nD|$ for $n \gg 0$.
  - Freedom here that is crucial: If $D = \sum_{i=1}^{r} D_i$, we can also look at maps associated to divisors $D' = \sum_{i=1}^{r} a_i D_i$, $a_i \in \mathbb{N}$.
  - Suppose the D_i are all ample.
  - Best choice: Choose $a_i \in \mathbb{N}$ such that if $D' = \sum_{i=1}^{r} a_i D_i$, in the embedding given by $|nD'|$, $n \gg 0$, the degrees of $a_1 D_1, \ldots a_r D_r$ are all equal (this can essentially be done).
- Finding a good set of hyperplanes $H_{0v}, \ldots, H_{Nv}$, for each $v \in S$.
  - This reduces to the following pure algebraic geometry problem: For every point $P \in D$, find a basis B of $L(nD)$ such that the basis vanishes "on average" at P.
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- Finding a good set of hyperplanes $H_{0v}, \ldots, H_{Nv}$, for each $v \in S$.
  - This reduces to the following pure algebraic geometry problem: For every point $P \in D$, find a basis B of $L(nD)$ such that the basis vanishes "on average" at P.
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The Corvaja-Zannier Subspace Theorem Method

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