Approximation algorithms and the hardness of approximation

Chandra Chekuri
(Dept. of Computer Science, University of Illinois at Urbana-Champaign, Urbana, IL),
Joseph Cheriyan
(Dept. of Combinatorics and Optimization, University of Waterloo, Waterloo, ON),
Ryan O'Donnell
(School of Computer Science, Carnegie Mellon University, Pittsburgh, PA),
Mohammad R. Salavatipour
(Dept. of Computing Science, University of Alberta, Edmonton, AL),
David P. Williamson
(School of Operations Research and Information Engineering, Cornell University, Ithaca, NY)

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1 Overview of the Field

Most of the many discrete optimization problems arising in the sciences, engineering, and mathematics are NP-hard, that is, there exist no efficient algorithms to solve them to optimality, assuming the P≠NP conjecture.[1] The area of approximation algorithms focuses on the design and analysis of efficient algorithms that find solutions that are within a guaranteed factor of the optimal one. Loosely speaking, in the context of studying algorithmic problems, an approximation guarantee captures the "goodness" of an algorithm – for every possible set of input data for the problem, the algorithm finds a solution whose cost is within this factor of the optimal cost. A hardness threshold indicates the "badness" of the algorithmic problem – no efficient algorithm can achieve an approximation guarantee better than the hardness threshold assuming that P≠NP (or a similar complexity assumption). Over the last two decades, there have been major advances on the design and analysis of approximation algorithms, and on the complementary topic of the hardness of approximation, see [45], [46].

The long-term agenda of our area (approximation algorithms and hardness results) is to classify all of the fundamental NP-hard problems according to their approximability and hardness thresholds. This agenda may seem far-fetched, but remarkable progress has been made over the last two decades. Approximation guarantees and hardness thresholds that "match" each other have been established for key problems in topics such as:

- covering and partitioning (the set covering problem [17]),
- algebra (overdetermined system of equations [21])
- graphs (clique, colouring [47]),

[1] This workshop is a sequel to our workshop 11w5117 on the same topic; some of the introductory/filler text is the same for both reports, but the actual contents are different.

- combinatorial optimization (maximum cut [18],[26]),
- constraint satisfaction (maximum satisfiability problems [18],[21]), etc.

2 Objectives of the Workshop Proposal

The goals of the workshop are as follows:

1. To bring together researchers in the fields of approximation algorithms (who work on finding algorithms with good approximation guarantees) and complexity theory (who work on finding hardness thresholds), and to stimulate the exchange of ideas and techniques between the two groups.
2. To focus on a few key topics that could lead to deep new results in the areas of approximation algorithms, combinatorial optimization, hardness of approximation, and proof complexity. We describe a few ambitious topics below.

   (a) The most famous problem in all of discrete optimization is the Traveling Salesman Problem (TSP). Yet despite the attention paid to this problem, its approximability remains poorly understood. The best known approximation algorithm for the symmetric case is a 3/2-approximation algorithm due to Christofides from 1976. On the other hand, the known hardness-of-approximation results are weak, and there is a substantial gap between upper bounds and lower bounds.

   Recently, there has been remarkable progress on some special cases of the TSP and on some variants of the TSP, but, as of now, there has been no improvement on the approximation guarantee of 3/2 for the TSP. Some of the recent advances were covered in the 2011 BIRS workshop, and a whole day was devoted to talks on the TSP and related problems. Many of these advances are based on new and beautiful connections with probability theory, coupled with technically difficult exploitation of methods and structures that are studied in combinatorial optimization. For example, Oveis Gharan, Saberi, and Singh [38] used properties of strongly Rayleigh measures coupled with an elaborate analysis of the structure of near-minimum cuts to obtain the first improvement on the 3/2-approximation guarantee for a key special case called the graphic TSP; also, see Asadpour et al. [7]. The most recent result on this special case is a 7/5-approximation algorithm of Sebő and Vygen [43] that hinges on a probabilistic lemma of Momke and Svensson [34] coupled with an in-depth and novel analysis of structures that are well known in combinatorial optimization. An, Kleinberg, and Shmoys [1] improved on a 20-year old 5/3-approximation guarantee of Hoogeveen [23] for the s-t path TSP, which is a variant of TSP; they use a randomized rounding algorithm, and their improvement uses probabilistic methods coupled with an analysis of near-minimum cuts. Very recently, Sebő [42] has improved on this result to obtain an 8/5-approximation guarantee, by using further probabilistic insights.

   It has long been conjectured that there is a 4/3-approximation algorithm for the TSP, and a 3/2-approximation algorithm for the s-t path TSP. By re-focusing attention on these conjectures, our goal is to continue the momentum from the 2011 BIRS workshop on the TSP.

   (b) Another focus topic will be approximation algorithms for disjoint paths and related routing problems. These problems are intimately related to fundamental topics on structural graph theory as well as to multicommodity flows. At the 2011 BIRS workshop, Chuzhoy [15] presented a plenary talk on her breakthrough work that obtained the first poly-logarithmic approximation algorithm with constant congestion for the maximum edge-disjoint paths problem (MEDP). This built on a long line of work and numerous tools. There has been subsequent exciting progress including the work of Chuzhoy and Li [16] who improved the congestion bound from 14 to 2.

   Despite this progress, the approximability of the maximum disjoint paths problem is wide open if the congestion bound is kept at one. Let n denote the number of nodes of the input graph. The known upper bound is $O(\sqrt{n})$ while the lower bound is only sub-logarithmic in n (namely, $\Omega(\log^{\frac{1}{2}-\epsilon} n)$), see Andrews et al. [3]. The approximability threshold is poorly understood even for restricted instances such as planar graphs and graphs of bounded treewidth. The impetus from the recent results and tools promises to lead to exciting new advances on algorithms for routing problems. The results of Chuzhoy [15] have been extended to the maximum node-disjoint paths problem with constant congestion by Chekuri and Ene [14]. In addition, there are strong connections to structural aspects of graph theory. The recent progress on disjoint path problems is based on proving the existence of routing structures whose size is proportional to the treewidth of the graph, see [14]. This points to further connections between the above algorithmic results and the theory of Graph Minors developed by Robertson and Seymour, [41].

Our aim is to start systematic explorations of this connection. On the one hand, this could lead to the design of powerful new algorithmic tools for some of the routing problems that arise in many applied areas, and on the other hand, it could provide new techniques for attacking some of the significant conjectures within the theory of Graph Minors.

(c) A key component of the workshop will focus on the notorious Unique Games Conjecture (UGC) and surrounding issues in the complexity of optimization problems. Since its proposal in 2002 by Khot [25], the UGC has led to major new (conditional) inapproximability results for constraint satisfaction problems (CSPs). This culminated in Raghavendra's 2009 work [40] giving a polynomial-time algorithm for optimally approximating each CSP, assuming the UGC.

The workshop will explore two directions that are at the vanguard of UGC research: improving optimization algorithms via Lasserre / sum-of-squares (SOS) proofs methodology, and going beyond the UGC in complexity results.

(*Lasserre / SOS algorithms.*) In 2011, it was shown that the powerful Lasserre SDP hierarchy of algorithms (see [29], and also see [44], [33]) could be used to obtain a subexponential-time algorithm for Unique Games (UG) (as in Arora et al. [4]) and some related problems, see Barak et al. [11] and Guruswami et al. [19]. (The 2011 BIRS workshop included two talks on these topics, by Georgiou and Steurer.) These results are motivating further advances in optimization using Lasserre algorithms.

A very recent, important work of Barak et al. [8] emphasized and applied the connection between Lasserre algorithms and Sum-of-Squares (SOS) proof complexity; they showed that the known "hard instances" of the UG problem can be analyzed by constant-degree SOS proofs, and hence solved by a polynomial-time algorithm. Subsequent research furthered the connection to proof complexity, see O'Donnell et al. [37]. The organizers are eager to bring together researchers working on optimization algorithms and on proof complexity, with the belief that these newly developed connections may lead to dramatic breakthroughs in efficient approximation algorithms.

(*Beyond the UGC.*) One emerging stream of current research is concerned with going beyond the UGC. In one direction, going beyond the UGC has involved "upgrading" known UG-hardness results to full-fledged NP-hardness results. New techniques in this area, such as "smooth Label Cover" (see Guruswami et al. [20] and Hastad [22]) and "PCPs robust against low-degree polynomials" (see Chan [12] and Khot et al. [28]), are giving hope that many optimization tasks known to be "UG-hard" can be proved to be NP-hard without needing to resolve the UGC.

In the other direction, going beyond the UGC involves introducing alternative conjectures which can lead to inapproximability results not provable via UGC; one example is the Projection Games Conjecture introduced at the 2011 BIRS workshop by Moshkovitz [35].

3. To include many younger researchers, and foster a relaxed interaction with established researchers. Our goal is to have a third of the workshop participants from this group. We expect a good number of female participants, similar to the workshop 11w5117. (There were five female participants.)

4. To allow groups of Canadian researchers working in this area to meet, and either initiate or renew collaborations.

3 Recent Developments and Open Problems

The study of approximation algorithms and the hardness of approximation is one of the most exciting areas among researchers in theoretical computer science; every major conference in the field has several papers on these topics. Significant progress is being made. We give a few examples of recent, dramatic innovations:

Arora, Barak and Steurer [4] recently presented a sub-exponential-time algorithm for Unique Games. This is a landmark result on one of the most perplexing questions in the area, namely the Unique Games Conjecture (UGC). While this result does not disprove the UGC, it gives strong indications that the UGC may not be true. The implication is that some of the remarkable tight approximation thresholds proved assuming the UGC need to be revisited; one possibility is that substantially deeper methods may be needed to prove the tightness of these approximation thresholds assuming the P≠NP conjecture.

Chuzhoy and Li [16] (FOCS 2012) recently presented an approximation algorithm for maximizing the number of edge disjoint paths with congestion two that achieves a poly-logarithmic approximation guarantee with respect to a standard LP relaxation. This is a remarkable achievement, because their LP relaxation has an integrality ratio that is much higher (than poly-logarithmic) for congestion one, meaning that poly-logarithmic approximation guarantees for congestion one are not possible (based on their LP relaxation); moreover, qualitatively speaking, the approximation guarantee for congestion two is best possible, due to lower bounds from the area of hardness of approximation.

Chekuri and Chuzhoy [13] (STOC 2014) very recently made substantial progress on an important graph theoretic problem. This was inspired by techniques from the recent work of Chuzhoy [15], Chuzhoy and Li [16] and Chekuri and Ene [14] on approximating the maximum disjoint paths problem. They showed that any graph G with treewidth at least $k^{98-o(1)}$ has a grid minor of size k. This establishes the first polynomial relationship between treewidth and the size of the largest grid-minor in a graph; previous bounds from the seminal work of Robertson and Seymour on graph minors and subsequent improvements required the treewidth of G to be an exponential function of k to ensure that G has a grid-minor of size k. This work is an example of the rich interplay between structural and algorithmic results.

4 Presentation Highlights

4.1 LP-Based Algorithms for Capacitated Facility Location

The first plenary talk was given by Hyung-Chan An (EPFL, Lausanne), on some recent advances obtained jointly with Mohit Singh and Ola Svensson, [2].

Linear programming has played a key role in the study of algorithms for combinatorial optimization problems. In the field of approximation algorithms, this is well illustrated by the uncapacitated facility location problem. A variety of algorithmic methodologies, such as LP-rounding and the primal-dual method have been applied to and evolved from algorithms for this problem. Unfortunately, this collection of powerful algorithmic techniques had not yet been applicable to the more general capacitated facility location problem. In fact, all of the known algorithms with good performance guarantees were based on a single technique, local search, and moreover, no linear programming relaxation was known to efficiently approximate the problem.

In their new paper, they present a linear programming relaxation with constant integrality gap for capacitated facility location. An et al. demonstrate that fundamental concepts from matching theory, including alternating paths and residual networks, provide key insights that lead to the strong relaxation. Their algorithmic proof of the integrality gap is obtained by finally accessing the rich toolbox of LP-based methodologies: they present a constant factor approximation algorithm based on LP-rounding. Their results resolve one of the ten open problems selected by the textbook on approximation algorithms of Williamson and Shmoys [46].

4.2 Open Questions in Parallel Repetition of Games and PCPs

Irit Dinur (Weizmann Institute) presented a plenary talk on a collection of questions that addresses parallel repetition and PCPs.

1. Parallel repetition of k player games ($k = 3$ or more): while we know a lot about the value of a repeated two player game, much less is known for k player games.

2. Direct sum of games: if parallel repetition is the direct product of games, then the direct sum operation is easy to define for XOR games. Some interesting things are known here, but no version of the XOR lemma has been proved, as of now.

3. Derandomized parallel repetition: What results can we expect to get if we disallow randomization? Is there a PCP theorem with polynomially small error?

4.3 On the Power of Symmetric LP/SDP Relaxations

Prasad Raghavendra (UC Berkeley) presented a plenary talk on two recent results obtained jointly with James Lee, David Steurer, and Ning Tan [31].

1. They show that for $k < n/4$, the k-rounds sum-of-squares or Lasserre SDP relaxation achieves the best possible approximation guarantee for Max-CSPs among all symmetric SDP relaxations of size at most $\binom{n}{k}$.
2. They show how to construct linear programs for TSP that are instance-optimal among all symmetric linear programs.

4.4 Sum-of-Squares Method, Tensor Decomposition, and Dictionary Learning

The final plenary talk was given by David Steurer (Cornell), and it presented recent results obtained jointly with Boaz Barak and Jonathan Kelner [10].

The sum-of-squares method is a widely-studied, conceptually simple meta-algorithm that, for a broad range of problems, captures the best known algorithms and potentially achieves better and plausibly optimal guarantees for these problems. Barak et al. introduce a general approach for proving guarantees of efficient approximation algorithms based on the sum-of-squares method by exploiting connections to proof complexity. Following this approach, they give efficient algorithms with significantly improved guarantees for several problems arising in machine learning and optimization, in particular, robust tensor decomposition and dictionary learning.

5 Scientific Progress Made and Outcome of the Meeting

The schedule of the workshop provided ample free time for participants to work on joint research projects. A number of new research projects were initiated during the workshop, while some other researchers used the opportunity to continue to work on projects started earlier. The research talks and the plenary talks were very well received.

Poloczek, Schnitger, Williamson, and van Zuylen gave a randomized greedy algorithm that obtains an expected 3/4-approximation for the maximum satisfiability problem. It remained open whether there is a deterministic algorithm that achieves the same approximation guarantee without using linear programming. After Poloczek's talk at the workshop, Svensson pointed out that such an algorithm can be obtained by derandomizing the fractional greedy algorithm of Buchbinder, Feldman, Naor, and Schwartz [9] using the method of conditional expectation. In the meantime, this algorithm was analyzed exactly and shown to run in linear time. Moreover, the algorithmic idea has been extended to further combinatorial optimization problems.

As a result of some discussions started during the workshop, Ene, Nguyen, and Vondrak can show that there is a $(k-1)$-approximation for the problem of partitioning a ground set into k pieces while minimizing a separation cost measured by a (possibly non-monotone, asymmetric) submodular function, plus additive assignment costs for element/label pairs. They can also show a Δ-approximation for this problem when the submodular function is given explicitly as a sum of predicates depending on Δ elements each. Both results match the hardness result that Vondrak presented in the workshop for Hypergraph Labeling, which is a special case of this problem.

Anupam Gupta, inspired by questions after his talk on greedy algorithms for the Steiner Forest problem, developed a cleaner presentation of the algorithm, with an improved analysis. He also started new collaborations with Chandra Chekuri on a generalization of the multi-way cut problem, and with Bruce Shepherd on the edge-disjoint-paths problem.

Participants James Lee, Prasad Raghavendra, and David Steurer used the workshop to restart their collaboration on characterizing the power of semidefinite programs (SDPs). Since then, they have introduced a method for proving lower bounds on the efficacy of SDP relaxations for combinatorial problems. In particular, they show that the cut, TSP, and stable set polytopes on n-vertex graphs are not the linear image of the feasible region of any SDP (i.e., any spectrahedron) of dimension less than 2^{n^c} for some constant $c > 0$. Their result yields the first super-polynomial lower bounds on the semidefinite extension complexity of any explicit family of polytopes.

Irit Dinur, David Steurer, and Prasad Raghavendra submitted a new joint grant application partly based on discussions started at the Banff workshop.

The above are only a few examples of the research progress made during or after the workshop, and there are other ongoing projects that started at the workshop.

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