Geometric Aspects of Semilinear Elliptic and Parabolic Equations: Recent Advances and Future Perspectives

Manuel del Pino (University of Chile),
Peter Poláčik (University of Minnesota),
Juncheng Wei (University of British Columbia)

May 25 - 30, 2014

1   Introduction

This five-day workshop provided a forum for the dissemination of current advances in geometric properties of solutions to semilinear elliptic and parabolic problems, and connections of these properties with minimal and constant mean curvature surfaces. Also, recent progress in the analysis of traveling-front solutions and blow-up patterns for reaction-diffusion equations was exposed in several lectures and discussed in connections with new results on eternal solutions and singularity formation for geometric flows.

The meeting brought together two groups of mathematicians. One working on partial differential equations (PDEs), especially in construction and classification of entire and eternal solutions of semilinear elliptic and parabolic equations, and a second group, in differential geometry, including specialists in constant mean curvature and minimal surface theories, and in geometric flows. The main goal of the workshop was to explore deeper the connections between these two fields, and inspire participating researchers into new developments, exposing them to methods and phenomena discovered separately in the geometric and the pure PDE sides.

2   Overview and Recent Developments in the Field

The study of entire and eternal solutions to semilinear elliptic and parabolic PDEs is an area of investigation with long tradition and the results obtained have typically far reaching consequences in other areas and models of higher complexity. For instance, the classification of solutions of elliptic equations and systems in bounded domains depends in a crucial way on the existence or nonexistence of entire solutions in the whole space or on the symmetry properties of these solutions. In both elliptic and parabolic equations, Liouville theorems (asserting the nonexistence of entire solutions of specific semilinear equations) provide a powerful tool for the study of singularities of solutions to rather general problems via scaling arguments.

A number of recent works have explored connections and analogies between constant mean curvature and minimal surface theories and patterns present on the level sets of families of entire solutions of semilinear elliptic equations. This is the case, for instance, with a counterexample in dimension 9 to a famous conjecture

by De Giorgi for solutions to the Allen Cahn equation of phase transitions by del Pino, Kowalczyk, Wei (2008). The solution found has level sets which nearly reproduce large dilations of a minimal surface, which is the graph of an entire function, found by Bombieri, De Giorgi and Giusti in 1969 as a counterexample in dimension 9 to Bernstein's conjecture. The correspondence between minimal surfaces and solutions of the Allen Cahn equation has major consequences in the understanding of its solution set (which also offers major differences!). Another important example which offers striking analogies with geometric objects is the standing-wave problem for the focusing nonlinear Schrödinger equation. Various constructions of new solutions have been achieved in recent years. In particular we can mention works by del Pino, Kowalczyk, Malchiodi, Musso, Pacard, Santra, Wei, based on methods which parallel gluing theories for construction of constant mean curvature (CMC) surfaces and Yamabe singular metrics by Kapouleas, Mazzeo, Pacard, Pollack, Schoen and other authors.

The natural time dependent analogues of those issues correspond to establishing connections between reaction-diffusion equations and geometric flows. For instance, a lot of interesting phenomena have been discovered in recent years in traveling-wave type solutions to reaction-diffusion equations of KPP or Allen Cahn types, which parallel eternal, self translating solutions to mean curvature type flows. A typical example is the construction of radially symmetric traveling waves by Chen, Guo, Hamel, Ninomiya, and Roquejoffre (2007), which parallels the eternal traveling solitons of mean curvature flows (XJ Wang, 2004). Many interesting traveling waves found in the unbalanced Allen-Cahn equations, including pyramids (Taniguchi 2008) parallel the traveling solitons in forced mean curvature flows. Some other new traveling waves are also found through the interplay of PDE and geometry (the Jacobi-Toda system in nested traveling waves by del Pino, Kowalczyk, Wei , 2010). New traveling solitons to the mean curvature flows have also been constructed recently (XJ Wang 2011, Monneau, Roquejoffre, Roussier-Michon 2011). Along the same lines, connections between singularity formation in geometric flows and blow-up in reaction diffusion equations is a related, fascinating matter, relatively less treated.

3   Presentation Highlights

In accord with the main goals of the meeting, the program included lectures by specialist in differential geometry and PDEs. Specifically, several lectures were devoted to the existence and properties of surfaces of constant mean curvature and similar geometric objects (Hauswirth, Kapouleas, Mazzeo, Nguyen, Valdinoci, K. Wang). Related lectures from the PDE side included studies of solutions of elliptic equations via variational methods and investigation of properties of the solutions by geometric and analytic techniques (Dancer, Gui, Ghoussoub, Hamel, Jerrard, Li, Lin, Liu, Pistoia, Robert, Z.-Q. Wang, Yan). Likewise, time-dependent problems were discussed by specialists in both groups. Geometric flows were examined in the lectures by Angenent, Daskalopoulos, Sesum, and several parallels of their results were found in PDE lectures by Fila, King, Souplet, Winkler, Yanagida, and others.

As mentioned above, Liouville and classification theorems play an important role in differential geometry as well as in PDEs. The lectures by Quittner and Terracini addressed the validity of Liouville theorems in various elliptic problems, whereas lectures by Angenent and Daskalopoulos included classification results for entire solutions of geometric flows. Traveling-wave type solutions, which are important examples of entire solutions to reaction-diffusion equations, were the main topic of the lectures by Cabre, Ninomiya, Taniguchi. Several connections to geometric problems were highlighted in the discussions to these lectures. A very lively and fruitful discussion was also spurred by lectures concerning singularities and blow up phenomena in parabolic equations (Fila, Souplet, Winkler, Yanagida) and geometric flows (Daskalopoulos, Sesum). In the lecture by Sire, a new proof of Kolmogorov-Arnold-Moser theorems in infinite dimensions was presented. The theorem applies to ill-posed equations, including elliptic equations on unbounded strips.

In the geometric side, classical research topics such as constant mean curvature surfaces and minimal surfaces are still very active with new ideas from algebraic and nonlinear PDEs (Mazzeo, Hauswirth, Nguyen). Geometric flow including curve shortening and Ricci and Yamabe flows also received much attention (Angenent, Daskalopoulos, Sesum). This interacts well with the study of parabolic blow-ups (King, Fila, Souplet, Yanagida, Winkler).

An interesting highlight was the Kelei Wang's new proof of Savin's Theorem for De Giorgi conjecture in dimensions less than 9. A hybrid of geometric methods in minimal surfaces and geometric measure theory and PDE techniques is used to give a more geometric proof of De Giorgi's conjecture.

4 Scientific Progress Made

The speakers brought many very interesting results to the conference. These were presented in their lectures and discussed right after or during times reserved for more informal discussions. Some of the theorems presented were brand new results, not yet available in the literature. The dissemination of the such results was one of the main goals of the conference. We also consider it a major achievement of the meeting that somewhat disjoint groups of scientists working on related problems exhanged their points of view. Combining strengths of various approaches will inavitably lead to progress in these research projects.

In the following we summarize the main points of individual talks. Also, if available, we provide references to publications containing detailed proofs of some results presented in the lecture; or we quote other papers of the lecturer where similar techniques have been used.

- **Constant mean curvature and similar geometric objects**

    - **Rafe Mazzeo** [32] considered the constant mean curvature sphere, and introduced a new definition of constant mean curvature for sub-manifolds of co-dimension greater than one. He proved existence of small sphere-like solutions of this problem which are tied to critical points of a certain curvature functional and also talk about a Ginzburg-Landau type relaxation of this problem in three dimensions.

    - **L. Hauswirth** [24, 17] gave a survey on the algebraic and geometric aspects of CMC surfaces. Among other things, he studied the CMC torus and CMC annulus in $\mathbb{R}^3$ and the relation with the integral system.

    - **E. Valdinoci** [6, 5, 9, 42] presented some recent results on the qualitative properties and the regularity theory for the minimizers of a nonlocal perimeter functional. He also discussed some related free boundary problems in relation with density estimates and homogeneous solutions.

    - **Xuan H. Nguyen** [36, 37, 38] considered the construction of new minimal surfaces by the technique of desingularization which has been used by Kapouleas and Traizet in 1990's. She mentioned some recent progress in this direction which involves interaction of immersed surfaces and also discuss new challenges.

    - **Kelei Wang** [47] gave a survey on the improvement of flatness type estimates in nonlinear problems. One main ingredient, which goes back to De Giorgi, is to approximate nonlinear problems by linear ones such as harmonic functions. He discusses the generalization of this classical method to singular perturbation problems of semilinear elliptic equations, and applications of these results to the De Giorgi conjecture for several entire space problems.

- **Studies of elliptic partial differential equation and time dependent problems**.

    - **E.N.Dancer** [11] survey some bifurcation results for positive solutions of the Dirichlet boundary problem $-\Delta u = r f(u)$ in $D \subset \mathbb{R}^n$ as well as some other boundary conditions when $n \geq 3$ and f grows faster than exponentially. Some open problems in $\mathbb{R}^2$ are also discussed.

    - **C. Gui** [19, 18] discussed the even symmetry in the direction of the axis for axially symmetric solutions of Allen-Cahn equation. His proof was based on the asymptotic behavior of nodal sets of such solutions and the moving plane method.

    - **Nassif Ghoussoub** [21] showed that the components $u = (u_1, \cdots, u_m) : R^N \to R^m$ of solutions of elliptic Allen-Cahn system $\Delta u = \nabla H(u)$ are also solutions of a much more tractable decoupled system of ODEs $\Delta u_i = V_i'(u_i(x))$ where $V_1, \cdots, V_m$ are appropriate potentials by Monge-Kantorovich optimal transport theory.

- **Francois Hamel** [23] provided some point-wise comparison results between the solutions of some second-order semilinear elliptic equations in a domain of R^n and the solutions of some radially symmetric equations in the equi-measurable ball. Both equations with linear growth in the gradient and equations with at most quadratic growth in the gradient are considered. His idea is based on a symmetrization of the second-order terms.

- **R. Jerrard** [2] talked about the application of Gamma-convergence which provide a way associating a limiting functional to a sequence of functionals E_ϵ such that sequences of minimizers of E_ϵ converge to a minimizer of the limiting functional. He started with the result which shows that, given a saddle point of the limiting functional, one can find associated critical points of E_ϵ and discussed some recent related developments.

- **Yanyan Li** [28] considered the semi-linear elliptic equation $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$ with positive periodic function $K(x)$ for $n > 2$. Under some natural condition on K, he proved the existence of multi-bump solutions centered on lattices in R^k if $1 < k < \frac{n-2}{2}$ and non-existence of such solutions for $k > \frac{n-2}{2}$.

- **Chang-Shou Lin** [8] considered the Liouville equation $\Delta u + e^u = \rho\delta_0$ in a two dimensional torus E_τ. He developed a theory which connect the Liouville equation, hyper-elliptic curves and modular forms. He talked about the development in this direction and also some open problems.

- **Yong Liu** [27, 26] studied the Allen-Cahn equation in dimension 3:

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^3.$$

He constructed new family of solutions whose behavior at infinity is well controlled: the nodal sets of these solutions outside of a large ball are of catenoid type. This is closely related to the minimal surface theory in dimension 3.

- **Robert McCann** [35] considered 2-Wasserstein gradient flows of the generalized Fisher information which are fourth order family of evolution equations including the thin-film and quantum drift-diffusion equations. He diagonalized this linearization of the equation by relating it to analogous problem for the porous medium equation and get information about the leading-and higher-order asymptotics of the fourth-order flows on R^n which were inaccessible previously.

- **Hirokazu Ninomiya** [34] showed the existence of the traveling spot including the front and the back for two-dimensional excitable media. Using the traveling spots, he also mentioned some mathematical understanding of the formation of spirals which was deeply related to the ventricular brillation.

- **Angela Pistoia** [40] considered the stationary Keller-Segel system from chemotaxis in a domain of the plane. She proves the existence of a solution which exhibits a sharp boundary layer along the entire boundary of the domain.

- **Pavol Quittner** [41] proved a priori estimates and existence of positive solutions of semilinear elliptic systems with power nonlinearities and homogeneous Dirichlet boundary conditions. He also talked about some new Liouville-type theorems.

- **F. Robert** [22] considered the existence of positive solutions for the Dirichlet boundary problem $\Delta u - \frac{\gamma}{|x|^2}u = \frac{u^{2*(s)-1}}{|x|^2}$ in $\Omega \subset \mathbb{R}^n$ where $0 \in \partial\Omega$. It is known that this is indeed the case when $\gamma < \frac{(n-2)^2}{4}$ under the condition that the mean curvature of the domain at 0 is negative. Robert proved that $\gamma = \frac{n^2-1}{4}$ is another critical threshold for the operator on the left of the equation.

- **Susanna Terracini** [44] considered the classification of entire solutions of the competitive elliptic system

$$\begin{cases} \Delta u_i = \sum_{j \neq i} u_j^2 u_i, \ i = 1, \cdots, k, \text{ in } \mathbb{R}^N \\ u_i > 0 \text{ in } \mathbb{R}^N \end{cases}$$

She showed that a bound on the growth of a positive solution imposes a bound on the number of components k of the solution itself. Furthermore, she can prove the 1-dimensional symmetry of the solutions satisfying suitable assumptions, extending known results which are available for $k = 2$. Her proof rests upon a blow-down analysis and on some monotonicity formula.

– **Yannick Sire** [16, 14] developed a method to construct some special classes of solutions for a priori ill-posed equations. The method consists in a Nash-Moser iteration suitably designed. Application to several PDEs coming from fluid mechanics were provided.

– **Wang Zhi-Qiang** [39] reported recent work on existence and multiplicity of positive vector solutions for nonlinear Schrodinger systems with mixed couplings. He showed symmetric breaking for least energy solutions in symmetric domains. Coexistence of synchronization and segregation was also explored.

– **Shusen Yan** [30, 31, 29] presented some results on the structure of the bubbling solutions for Chern-Simons model in torus. The results include the co-existence of bubbles, and the existence of a one to one map between the bubbling solutions and non-degenerate critical points of some functions. By elliptic function theories, he also discussed the number of the critical points of those functions and their non-degeneracy.

• **Geometric flows and parallel PDE lectures**.

– **Sigurd Angenent** [1] considered the self similar solutions for curve shortening problem. Since compact self similar solutions for curve shortening are all planar and have been classified by Abresch-Langer. Angenent talked about the non compact self similar solutions.

– **Xavier Cabre** [3, 4] discussed some recent works on front propagation in the presence of fractional diffusion. First he proved the non-existence of traveling waves for the Fisher-KPP equation with fractional diffusion, both in homogeneous and in periodic media by establishing the exponential in time propagation of fronts. He also talks about the traveling fronts for the classical homogeneous heat equation in a half-plane with a boundary Neumann condition of bistable or combustion type.

– **P. Daskalopoulos** [12, 13] discussed the classification of ancient solutions to geometric flows such as the Ricci flow on surfaces and the Yamabe flow and also the construction of new ancient solutions to Yamabe flows by gluing methods.

– **Natasa Sesum** [33, 43] showed that the warped Berger solutions $(S^1 \times S^3, G(t))$ of Ricci flow develop finite-time neck-pinch singularities, and that they asymptotically approach round product metrics in space-time neighborhoods of their singular sets. This gave the first concrete example of Ricci flow solutions without rotational symmetry that develop locally finite-time neck pinch singularities.

– **M. Fila** [15] studied the asymptotic behaviour near extinction of positive solutions of the Cauchy problem for the fast diffusion equation with subcritical exponent. He showed the stability of separable solutions by finding a class of functions which belong to their domain of attraction. He also established optimal rates of convergence to separable solutions for such functions.

– **John King** [25, 15] studied the intermediate-asymptotic behaviour of a number of initial-value problems for the power-law nonlinear diffusion equation in which the critical exponents play important role. Related issues for the higher-order nonlinear diffusion were also mentioned

– **P. Souplet** [20] studied the quenching behavior for a semilinear heat equation arising in models of micro-electro mechanical systems. The problem involves a source term with a spatially dependent potential, given by the dielectric permittivity profile, and quenching corresponds to a touchdown phenomenon. He proved that touchdown cannot occur at zero points of the permittivity profile without the assumption of compactness of the touchdown set, made in all previous work on the subject and whose validity is unknown in most typical cases. This answered affirmatively a conjecture made in [Y. Guo, Z. Pan and M.J. Ward, SIAM J.Appl. Math 66 (2005), 309338] on the basis of numerical evidence. The result depends on a new type I estimate of the quenching rate. He mentioned that these results may be of some qualitative importance in applications to MEMS optimal design, especially for devices such as micro-valves.

– **M. Taniguchi** [45] studied traveling fronts to cooperative diffusion systems in R^N for $N \geq 3$. In particular, he proved the existence of traveling fronts associated with a $(N-2)$-dimensional

smooth surfaces which are boundaries of strictly convex compact sets in R^{N-1} and it is asymptotically stable for given initial perturbation. Moreover, he showed that the associated traveling fronts coincide up to phase transition if and only if the given surfaces satisfy an equivalence relation.

– **Michael Winkler** [48] studied the radially symmetric finite time blow up solutions to the Neumann initial-boundary value problem for the fully parabolic Keller-Segel system in a ball. He showed the existence of initial data such that finite time blow up occurs for $n \geq 3$ and also that the set of such blow-up enforcing initial data indeed is dense with respect to the topology of $L^p(\Omega) \times W^{1,2}(\Omega)$ for $p \in (1, \frac{2n}{n+2})$. The method of these results is based a more elaborate use of the natural energy inequality associated with the Kelley-Segel system.

– **Eiji Yanagida** [46] considered the time-dependent singular solutions for parabolic partial differential equations. He gave a sufficient condition for the removability of such singularities and also showed the existence of solutions with non-removable singularities.

5 Outcome of the Meeting

The workshop brought together two groups of mathematicians. One working on PDEs, especially in construction and classification of entire and eternal solutions of semilinear elliptic and parabolic equations, and a second group, in differential geometry, specialists in constant mean curvature and minimal surface theories, and in geometric flows. This workshop provided a good chance for these two groups of researchers to meet and interact with each other. It gave many opportunities to continue existing collaborations and to stimulate new mathematical ideas. Many participants saw new ideas arising from interaction with others who brought their own perspectives. A number of new collaborations appear to have arisen from the meeting, and the work of other collaborations was advanced. The large number of open problems that were presented in the talks will certainly encourage research in the area by participants and their Ph.D. students or Postdoctoral researchers.

References

[1] D.J. Altschuler, S.J. Altschuler, S.B. Angenent and L.F. Wu, The zoo of solitons for curve shortening in $\mathbb{R}^n$, *Nonlinearity* **26** (2013), no. 5, 11891226.

[2] S. Baldo, R.L. Jerrard, G. Orlandi and H.M. Soner, Convergence of Ginzburg-Landau functionals in three-dimensional superconductivity. *Arch. Ration. Mech. Anal.* **205** (2012), no. 3, 699752.

[3] X. Cabr, A.C. Coulon and J.M. Roquejoffre, Propagation in Fisher-KPP type equations with fractional diffusion in periodic media. *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 19-20, 885890.

[4] X. Cabr, N. Cnsul and J. V. Mand, Traveling wave solutions in a half-space for boundary reactions. *Anal. PDE* **8** (2015), no. 2, 333364.

[5] L. Caffarelli, O. Savin and E. Valdinoci, Minimization of a fractional perimeter-Dirichlet integral functional. *Ann. Inst. H. Poincar Anal. Non Linaire* **32** (2015), no. 4, 901924.

[6] L. Caffarelli and E. Valdinoci, Regularity properties of nonlocal minimal surfaces via limiting arguments. *Adv. Math.* **248** (2013), 843871.

[7] Y.Y. Chen, Y. Kohsaka and H. Ninomiya, Traveling spots and traveling fingers in singular limit problems of reaction-diffusion systems. *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), no. 3, 697714.

[8] C.L. Chai, C.S. Lin and C.L. Wang, Mean field equations, hyper-elliptic curves and modular forms: I. *Camb. J. Math.* **3** (2015), no. 1-2, 127274.

[9] A. Di Castro, M. Novaga, R. Matteo and E. Valdinoci, Nonlocal quantitative isoperimetric inequalities. *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 24212464.

[10] Y. Chen, Y. Kohsaka, and H. Ninomiya, *Traveling spots and traveling fingers in singular limit problems of reaction-diffusion systems*, Discrete Contin. Dyn. Syst. Ser. B **19** (2014), no. 3, 697–714.

[11] E.N. Dancer, Some bifurcation results for rapidly growing nonlinearities. *Discrete Contin. Dyn. Syst.* **33** (2013), no. 1, 153161.

[12] P. Daskalopoulos, R. Hamilton and N. Sesum, Classification of ancient compact solutions to the Ricci flow on surfaces. *J. Differential Geom.* **91** (2012), no. 2, 171214.

[13] P. Daskalopoulos, Manuel del Pino and N. Sesum, Type II ancient compact solutions to the Yamabe flow, *Journal fr die reine und angewandte Mathematik (Crelles Journal)*. ISSN (Online) 1435-5345, ISSN (Print) 0075-4102, DOI: 10.1515/crelle-2015-0048, October 2015.

[14] R. de la Llave and Y. Sire, *Kam theorem for whiskered tori in Hamiltonian partial differential equations*, **259** (in preparation), no. 6, 2180–2279.

[15] M. Fila, J.R. King and M. Winkler, Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent. *J. Lond. Math. Soc.* (2) **90** (2014), no. 1, 167183.

[16] E. Fontich, R. de la Llave, and Y. Sire, *Construction of invariant whiskered tori by a parameterization method. Part II: Quasi-periodic and almost periodic breathers in coupled map lattices*, J. Differential Equations **259** (2015), no. 6, 2180–2279.

[17] J. Gálvez, L. Hauswirth, and P. Mira, *Surfaces of constant curvature in $\mathbb{R}^3$ with isolated singularities*, Adv. Math. **241** (2013), 103–126.

[18] C. Gui, *Symmetry of traveling wave solutions to the Allen-Cahn equation in $\mathbb{R}^2$*, Arch. Ration. Mech. Anal. **203** (2012), no. 3, 1037–1065.

[19] C. Gui and F. Zhang, *Symmetry of entire solutions to the Allen-Cahn equation*, Adv. Nonlinear Stud. **15** (2015), no. 3, 587–612.

[20] J.S. Guo and P. Souplet, No touchdown at zero points of the permittivity profile for the MEMS problem. *SIAM J. Math. Anal.* **47** (2015), no. 1, 614625.

[21] N. Ghoussoub and B. Pass, Decoupling of DeGiorgi-type systems via multi-marginal optimal transport. *Comm. Partial Differential Equations* **39** (2014), no. 6, 10321047.

[22] N. Ghoussoub and F. Robert, The effect of curvature on the best constant in the Hardy-Sobolev inequalities. *Geom. Funct. Anal.* **16** (2006), no. 6, 12011245.

[23] F. Hamel, N. Nadirashvili and E. Russ, Rearrangement inequalities and applications to isoperimetric problems for eigenvalues. *Ann. of Math.* (2) **174** (2011), no. 2, 647755.

[24] L. Hauswirth, M. Kilian, and M. U. Schmidt, *Finite type minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$*, Illinois J. Math. **57** (2013), no. 3, 697–741.

[25] J. R. King, *Asymptotic analysis of extinction behaviour in fast nonlinear diffusion*, J. Engrg. Math. **66** (2010), no. 1-3, 65–86.

[26] M. Kowalczyk, Y. Liu, and F. Pacard, *The space of 4-ended solutions to the Allen-Cahn equation in the plane*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), no. 5, 761–781.

[27] M. Kowalczyk, Y. Liu, and F. Pacard, *The classification of four-end solutions to the Allen-Cahn equation on the plane*, Anal. PDE **6** (2013), no. 7, 1675–1718.

[28] Y.Y. Li, J.C. Wei and H. Xu, Multi-bump solutions of $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$ on Lattices in $\mathbb{R}^n$, *Journal fur die reine und ang. Math.*, to appear.

[29] C.-S. Lin and S. Yan, *Bubbling solutions for relativistic abelian Chern-Simons model on a torus*, Comm. Math. Phys. **297** (2010), no. 3, 733–758.

[30] C.-S. Lin and S. Yan, *A minimization problem associated with the Chern-Simons model with double vortex points on a torus*, Bull. Inst. Math. Acad. Sin. (N.S.) **8** (2013), no. 4, 491–503.

[31] C.-S. Lin and S. Yan, *Existence of bubbling solutions for Chern-Simons model on a torus*, Arch. Ration. Mech. Anal. **207** (2013), no. 2, 353–392.

[32] R. Mazzeo and F. Pacard, *Constant curvature foliations in asymptotically hyperbolic spaces*, Rev. Mat. Iberoam. **27** (2011), no. 1, 303–333.

[33] Rafe Mazzeo, Yanir A. Rubinstein, and Natasa Sesum, *Ricci flow on surfaces with conic singularities*, Anal. PDE **8** (2015), no. 4, 839–882.

[34] Y. Morita and H. Ninomiya, *Traveling wave solutions and entire solutions to reaction-diffusion equations [translation of mr2354291]*, Sugaku Expositions **23** (2010), no. 2, 213–233, Sugaku Expositions.

[35] R. J. McCann and C. Seis, The spectrum of a family of fourth-order nonlinear diffusions near the global attractor, *Comm. Partial Differential Equations* **40** (2015), no. 2, 191218.

[36] X. H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. I*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 1683–1701.

[37] X. H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. II*, Adv. Differential Equations **15** (2010), no. 5-6, 503–530.

[38] X. H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow, Part III*, Duke Math. J. **163** (2014), no. 11, 2023–2056.

[39] S.J. Peng and Z.Q. Wang, Segregated and synchronized vector solutions for nonlinear Schrdinger systems. *Arch. Ration. Mech. Anal.* **208** (2013), no. 1, 305339.

[40] A. Pistoia and G. Vaira, *Steady states with unbounded mass of the Keller-Segel system*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 1, 203–222.

[41] P. Quittner, A priori estimates, existence and Liouville theorems for semilinear elliptic systems with power nonlinearities. *Nonlinear Anal.* **102** (2014), 144158.

[42] O. Savin and E. Valdinoci, Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations* **48** (2013), no. 1-2, 3339.

[43] J. Isenberg, R. Mazzeo, and N. Sesum, *Ricci flow on asymptotically conical surfaces with nontrivial topology*, J. Reine Angew. Math. **676** (2013), 227–248.

[44] N. Soave and S. Terracini, Liouville theorems and 1-dimensional symmetry for solutions of an elliptic system modelling phase separation. *Adv. Math.* **279** (2015), 2966.

[45] M. Taniguchi, An (N1)-dimensional convex compact set gives an N-dimensional traveling front in the Allen-Cahn equation. *SIAM J. Math. Anal.* **47** (2015), no. 1, 455476.

[46] J. Takahashi and E. Yanagida, Time-dependent singularities in the heat equation. *Commun. Pure Appl. Anal.* **14** (2015), no. 3, 969979.

[47] K.L. Wang, Harmonic approximation and improvement of flatness in a singular perturbation problem. *Manuscripta Math.* **146** (2015), no. 1-2, 281298.

[48] M. Winkler, *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system*, J. Math. Pures Appl. (9) **100** (2013), no. 5, 748–767.