

53rd Cascade Topology Seminar

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1 Overview of the Field

Topology is a pervasive subject, with topological ideas arising in almost all branches of mathematics. As a discipline, topology is divided into several branches, including algebraic topology, geometric topology and differential topology. These branches can be very diffuse, with very little in common between different sub-disciplines, and indeed each branch can be very widely represented amongst many areas of specialization. The goal of the Cascade Topology Seminar is to establish some common ground between topologists in the Cascade region, including the Pacific Northwest United States and Western Canada.

At this particular meeting, there was a focus on algebraic topology. Within this subdiscipline, particular attention was given to the relationship between algebraic topology and K -theory and algebraic geometry, applied topology, and homotopy theory. **K -theory** is an invariant of categories with symmetric monoidal products, and it is in some sense a universal invariant because it is precisely the closest additive approximation of a symmetric monoidal category. K -theory has several incarnations: algebraic and topological K -theory may be the most frequently mentioned types, but K -theory can be widely applied. Indeed, many early applications of K -theory involved connections between topology and number theory, and in recent years more and diverse applications have been developed, especially to motivic cohomology and algebraic geometry. **Applied Topology** can refer to a number of emerging relations between topology and statistics and computer science. One of the most thriving areas of applied topology involves the relationship between topology and the study of data. Topology is the study of shape, and in particular topology is well-suited to study the shape of data. Algebraic invariants of topological spaces can be used to distinguish interesting features of data sets, and this has been a particularly fruitful method of data analysis because topology is well-suited to studying high-dimensional spaces, and algebraic tools are insensitive to small perturbations of data. **Homotopy Theory** is the study of topological spaces up to continuous deformations or homotopies. A central question in this field has been the discovery and computation of homotopy invariants which can determine whether two topological spaces are distinct or not.

2 Recent Developments and Open Problems

K -theory and algebraic geometry In the 1990's Voevodsky developed a program which brought together homotopy theory and algebraic geometry. In particular, this involved a program for doing homotopy theory for schemes, as well as developing motivic cohomology, an invariant for schemes. The latter brings algebraic K -theory into the picture, as motivic cohomology was used to prove Minor's conjecture relating the K -theory of a field to its étale cohomology. Voevodsky also proved the Bloch-Kato conjecture, which relates Milnor K -theory to Galois cohomology. Developing this program and related problems (such as computations) is a very active area of research involving many researchers examining different aspects of the program.

Applied Topology The chief tool of topological data analysis is persistent homology, which measures the topological features of simplicial complexes associated to a data set. These complexes vary over time from very coarse to very fine approximations of data, and persistent homology detects those features which are more important to the data because they persist throughout this process for long periods of time. Applications of persistent homology are emerging almost daily, as this tool is relatively new and ripe enough to be widely used. Medicine, astrophysics, computer science and engineering are a very few of the areas which have benefitted from topological data analysis. Theoretical developments in the field are often driven by applications which demand more features or precision, and extensions of persistent homology and metrics for the associated modules abound.

Homotopy Invariants An important, classical homotopy invariant of spaces is the Lusternik-Schnirelmann (LS-) category. The LS-category of X is the smallest integer n such that there is an open covering of X by n open sets, each of which is contractible in X . When applied to a manifold, the LS-category gives a bound on the number of critical points of a smooth function on the manifold. In 1971, Ganea conjectured that $cat(X \times S^n) = cat(X) + 1$ for all $n > 0$. Although many cases of this conjecture were demonstrated, in 1998 Iwase provided the first counterexample to Ganea's conjecture. For which spaces Ganea's conjecture holds remains an open question in this area, and bounds on $cat(X)$ in terms of related invariants are still being discovered.

3 Presentation Highlights

K -theory and algebraic geometry Four of the six talks were concentrated roughly in this area, though quite disperses within the field, ranging from geometric, to computational, to foundational. **Thritang Tran** offered an explanation of the homological stability of symmetric complements. A sequence of spaces is homologically stable if, eventually, the homology groups $H_i(X_k)$ do not depend on k . In this case, Dr. Tran examined certain explicitly constructed subspaces of symmetric powers of a space M . Dr. Tran explained how this provides a proof of Conjecture F of [?], which in turn is related to motivic stability and the stability in the Grothendieck ring of varieties.

The Farrell-Jones conjecture is the statement that the K - and L -theory of the group ring $R[G]$ is determined by group homology and the K - or L -theory of the group rings $R[V]$, where V varies over the virtually cyclic subgroups of G . The conjecture has been verified for a number of kinds of groups, including [?] and [?]. **Henrik Ruping** gave an overview of the Farrell-Jones conjectures, recent progress and applications.

Rick Jardine gave an overview of two approaches to homotopy theory for presheaves [?]. On the one hand, pro objects of simplicial presheaves is a model for traditional étale homotopy theory. On the other hand, cocycles give an efficient description of the homotopy category of simplicial presheaves. Dr. Jardine gave an overview of both of these approaches, and then gave insight into work in progress relating these two approaches. This subject matter forms part of the foundation of motivic homotopy theory and topological modular forms in stable homotopy theory.

To an audience of homotopy theorists, a *motive* (in the sense of e.g. Voevodsky or Deligne) is something like the stable homotopy category for algebraic geometry. **Jack Morava** offered a philosophy for trying to consolidate the various kinds of motives into a single concept, and explained some of the barriers that have prevented this from happening in the past. In particular, he conjectured the existence of a category which would be the analogue of mixed Tate motives, but built from $K(S^0)$ rather than $K(\mathbb{Z})$, and provided justification for this conjecture. This is inspired by work of Blumberg, Gepner, and Tabuada, who construct categories of non-commutative motives using small stable ∞ -categories.

Applied Topology Persistence modules, the time-graded homology modules associated to persistent homology, can be used to classify data. To do so, one needs to measure the distance between two persistence modules. One metric for doing so is the bottleneck metric. **Michael Lesnick** explained how the bottleneck metric can be generalized to incorporate multi-dimensional persistence (a generalization of persistence which measures data using multiple filtrations) by using interleavings. Published results appear in [?] and (now) [?].

Homotopy invariants **Don Stanley** presented recent progress on work on the LS-category of spaces which are products. It is well-known that $cat(X \times Y) \leq cat(X) + cat(Y)$. An open question is when equality is attained. Taking $Y = S^n$ relates this problem to Ganea's conjecture. Dr. Stanley gave a brief history of counterexamples to Ganea's conjecture and related constructions. In particular, he explained the relationship

between counterexamples to Ganea’s conjecture, and spaces with the property that $Qcat(X) < cat(X)$, where $Qcat$ is a kind of stabilized version of the LS-category. He demonstrated how this leads to the smallest dimensional known counterexample to Ganea’s conjecture [?], and Ordóñez and Stanley conjecture that this counterexamples is minimal.

4 Outcome of the Meeting

A theme of this meeting is that the majority of the talks included a broad overview of the particular topic or result to be covered. This is in keeping with the goal of the Cascade Topology Seminar, which hopes to build connections amongst the various kinds of topologists. In this meeting, not only were connections made across the subfields of topology, but also across career stages. The meeting included two graduate students, four post doctoral fellows and eleven faculty members.

References

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